Produkti in koprodukti družin

(za nazaj, spada pod družine množic)

Družina množic $A : I \to \mathrm{Set}$.

Unija $\bigcup A = \{ x \mid \exists i \in I .\ x \in A_i \}$ (množica)

Presek $\bigcap A = \{ x \mid \forall i \in I .\ x \in A_i \}$

Kartezični produkt

<u>Def</u>: <u>Funkcija izbire</u> za družino $A : I \to \mathrm{Set}$ je preslikava $f : I \to \bigcup A$, za katero velja $\forall i \in I .\ f(i) \in A_i$

<u>Primer</u>: $A : \mathbb{N} \to \mathrm{Set}$

$A_n := \{ x \in \mathbb{R} \mid 0 < x < 2^{-n} \} = (0, 2^{-n})$

Primer funkcije izbire za A je

$f : \mathbb{N} \to (0,1)$

$n \mapsto 2^{-n-1}$ ali $n \mapsto 2^{-n-2}$

ali $n \mapsto 3^{-n-1}$

<u>Primer</u>:

$B : \mathbb{N} \to \mathrm{Set}$

$B_n := \{ k \in \mathbb{N} \mid \overbrace{n \mid k}^{\exists j \in \mathbb{N} .\ k = nj} \}$

$B_3 = \{0, 3, 6, 9, 12, \dots\}$

$B_1 = \mathbb{N}$

$B_0 = \{0\}$

$f : \mathbb{N} \to \mathbb{N}$

$f : n \mapsto n$ ali $n \mapsto n^2$

ali $n \mapsto 0$

Primer:

$$C : \mathbb{R} \to \mathrm{Set} \qquad C_{-2} = \emptyset$$

$$C_x := \{ y \in \mathbb{R} \mid 0 < y < x \} = (0, x)$$

Funkcija izbire:

iščem $h : \mathbb{R} \to \mathbb{R}^+$ (pozitivna) da velja $h(x) \in C_x$ za $x \in \mathbb{R}$.

Funkcije izbire ni, ker je C_{-2} prazna množica:

če bi imeli h funkcijo izbire, bi veljalo $h(-2) \in C_{-2} = \emptyset$, kar ni možno

Def: Kartezični produkt družine $A : I \to \mathrm{Set}$ je množica $\prod A$ ali $\prod_{i \in I} A_i$, katere elementi so funkcije izbire za A:

$$\prod A := \{ f : I \to \bigcup A \mid \forall i \in I .\, f(i) \in A_i \}$$

Za vsak $j \in I$, definiramo j-to projekcijo

$$\pi_j : \prod A \to A_j$$
$$f \mapsto f(j)$$

Primer: $X \times Y$ je poseben primer:

$$A : \{0, 1\} \to \mathrm{Set}$$
$$A_0 := X$$
$$A_1 := Y$$

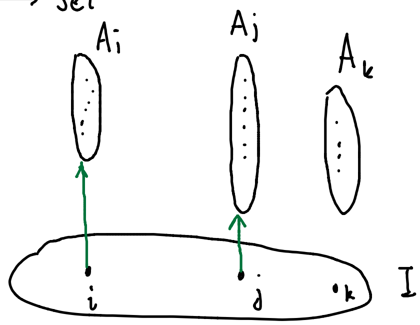
funkcija izbire $f : \{0,1\} \to X \cup Y$

$f(0) \in A_0 = X$

$f(1) \in A_1 = Y$

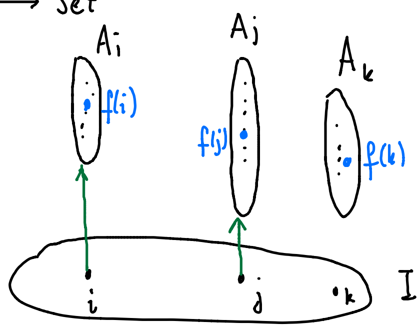
$$\prod A \cong X \times Y$$
$$f \mapsto (f(0), f(1))$$
$$\left(b \mapsto \begin{cases} x & b = 0, \\ y & b = 1 \end{cases} \right) \leftarrow (x, y)$$

__Slikice:__ $A : I \longrightarrow Set$

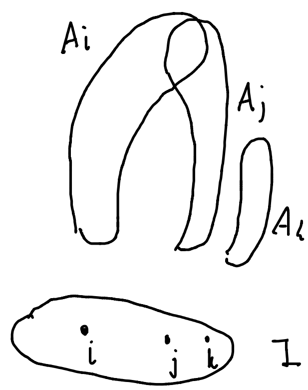


__Funkcija izbire:__

$A : I \longrightarrow Set$



__Unija družine:__



Vse to zgoraj zmetano na kup

__Def:__ __Koprodukt__ (__vsota__) družine $A : I \to Set$ je množica $\coprod A$, ali $\coprod_{i \in I} A_i$, z elementi $\underbrace{\iota_k}(x)$, kjer $k \in I$ in $x \in A_k$

$\iota_k : A_k \longrightarrow \coprod A$

kanonična injekcija

__Primer:__ Množici X in Y.

$A : \{0,1\} \longrightarrow Set$ $\qquad A_0 := X$, $\quad A_1 := Y$

$\coprod A$ elementi $\iota_k(z)$, kjer $k \in \{0,1\}$ in $z \in A_k$

Torej: $\iota_0(x)$ kjer $x \in A_0 = X$

$\iota_1(y)$ kjer $y \in A_1 = Y$

Ugotovili smo $\coprod A = X + Y$.

<u>Primer:</u> <u>Konstantna družina</u>

$$A : I \longrightarrow \mathrm{Set}, \quad S \in \mathrm{Set}$$

$$A_i = S \quad \text{za vse } i \in I$$

(z drugimi besedami: $A_i = A_j$ za $i, j \in I$)

Če je $I \neq \emptyset$:

$$\prod A = \{ f : I \to \overset{S}{\overset{\shortparallel}{\bigcup A}} \mid \forall i \in I .\, f(i) \in \overset{S}{\overset{\shortparallel}{A_i}} \}$$

$$= \{ f : I \to S \mid \underbrace{\forall i \in I .\, f(i) \in S}_{\top} \}$$

$$= S^I$$

$$\coprod A = \{ \iota_k(x) \mid k \in I, x \in \overset{S}{\overset{\shortparallel}{A_k}} \} = \{ \iota_k(x) \mid k \in I \text{ in } x \in S \}$$

$$\cong I \times S$$

$$\coprod A \cong I \times S$$

$$\iota_k(x) \longmapsto (k, x)$$

$$\iota_i(y) \longleftarrow\!\shortmid (i, y)$$

Premisli: $I = \emptyset$.

<u>Analiza:</u>

konstantno zaporedje $a : \{1, 2, 3, \dots, n\} \longrightarrow \mathbb{R}$ $\qquad s \in \mathbb{R}$

$a_i = s$

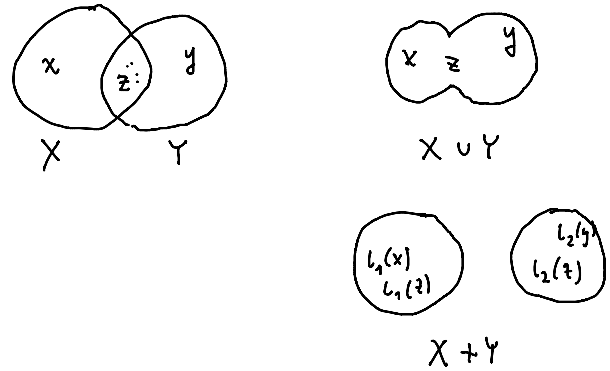
$$\sum_{i=1}^{n} a_i = \underbrace{s + s + s + \dots + s}_{n} = n \cdot s$$

$a : \{0, 1\} \to \mathbb{R}$

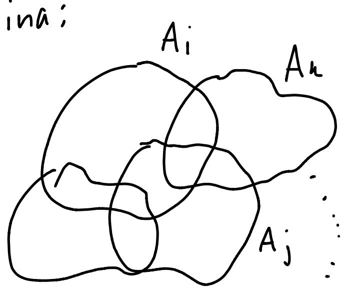
$a_0 = x$, $a_1 = y$

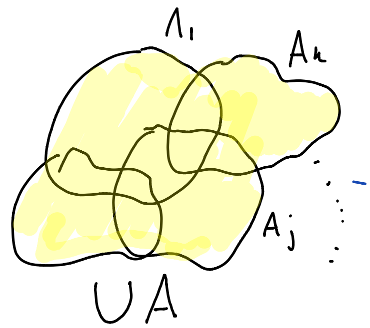
$$\sum_{i=0}^{1} a_i = a_0 + a_1 = x + y$$

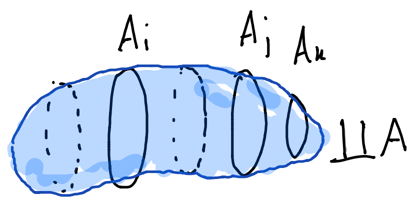
Slikice:



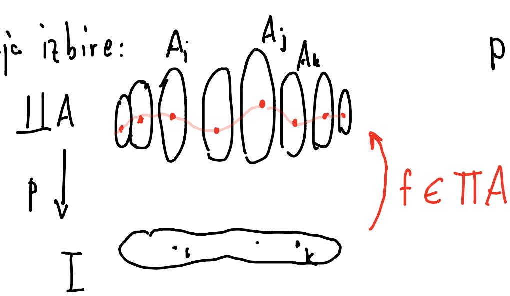
Družina:







Funkcije izbire:



$p : \coprod A \longrightarrow I$

$\iota_k(x) \longmapsto k$

Lastnosti preslikav II

Slike & praslike

Oznaka: Naj bo $f: A \to B$ preslikava.

Izpeljana množica: $\{f(x) \mid x \in A\} := \{y \in B \mid \exists x \in A . f(x) = y\}$

zapis podmnožice B

Izpeljana množica s pogojem:

$$\{f(x) \mid x \in A \mid \varphi(x)\} := \{y \in B \mid \exists x \in A . \varphi(x) \wedge f(x) = y\}$$

Običajno: $\{f(x) \mid x \in A \wedge \varphi(x)\}$

Primeri:

- Množica vseh popolnih kvadratov $\{n^2 \mid n \in \mathbb{N}\} = \{k \in \mathbb{N} \mid \exists n \in \mathbb{N} . k = n^2\}$
- Množica vseh popolnih kvadratov deljivih s 4:

$$\{n^2 \mid n \in \mathbb{N} \wedge 4 \mid n^2\} =$$

$$\{4k^2 \mid k \in \mathbb{N}\}$$

- $\{\sin(\frac{\pi k}{4}) \mid k \in \mathbb{N}\} = \{\sin 0, \sin\frac{\pi}{4}, \sin\frac{2\pi}{4}, \sin\frac{3\pi}{4}, \sin\frac{4\pi}{4}, \ldots\}$
$= \{0, \frac{\sqrt{2}}{2}, 1, -\frac{\sqrt{2}}{2}, -1\}$

Def: Naj bo $f: A \to B$ preslikava.

1. Praslika podmnožice $S \subseteq B$ je $f^*(S) := \{x \in A \mid f(x) \in S\}$
2. Slika podmnožice $T \subseteq A$ je $f_*(T) := \{f(x) \mid x \in T\} = \{y \in B \mid \exists x \in T . f(x) = y\}$

Zaloga vrednosti preslikave f je $f_*(A)$.

Pogosta (slaba!) oznaka za prasliko: $f^{-1}(S)$ f^{-1} inverz

$f^{-1}(U)$, $f^{-1}(x)$

Pogosta (slaba!) oznaka za sliko: $f(S)$ $f(\mathcal{B})$ slika ali f?

<u>Razmislek:</u> Dana je $f : A \to B$.

Praslika je preslikava:

$$f^* : P(B) \to P(A)$$
$$S \mapsto \{x \in A \mid fx \in S\}$$

$$f_* : P(A) \to P(B)$$

f surjektivna $\iff f_*(A) = B$

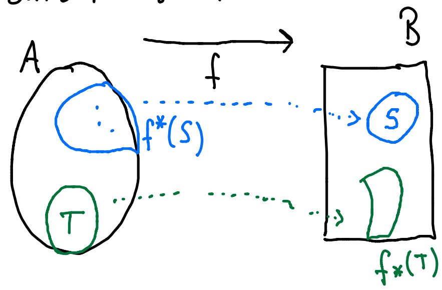
$$\square^* : B^A \to P(A)^{P(B)}$$
$$f \mapsto f^*$$

$$\square_* : B^A \to P(B)^{P(A)}$$

Sinonim:

praslika = inverzna slika

slika = direktna slika



Lastnosti f^* in f_* in $\cap$, $\cup$:

$f: A \to B$

$$f^*(T_1 \cup T_2) = f^*(T_1) \cup f^*(T_2) \quad \checkmark$$

$$\cap \qquad\qquad \cap \quad \checkmark$$

$$f^*(\emptyset) = \emptyset \quad \checkmark$$

$$f^*(B) = A \quad \checkmark$$

$$f^*\Big(\bigcup_{i \in I} T_i\Big) = \bigcup_{i \in I} f^*(T_i) \quad \checkmark$$

$T: I \to P(B)$

družina podmnožic B

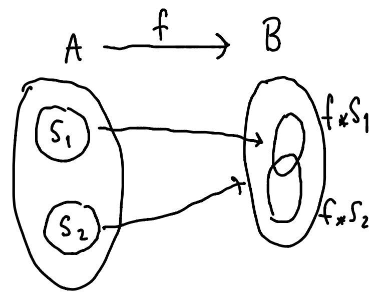
$$\bigcap \qquad\qquad \bigcap \quad \checkmark$$

$$f_*(S_1 \cup S_2) = f_*(S_1) \cup f_*(S_2)$$

$$f_*(\emptyset) = \emptyset$$

$$f_*(S_1 \cap S_2) \subseteq f_*(S_1) \cap f_*(S_2)$$

(= velja, če f injektivna)



$$f_*(A) \subseteq B$$

(= velja, če f surjektivna)