Merljivi kardinali

Mera:

1) Množica X
2) σ-algebra $\Sigma \subseteq P(X)$
3) mera $\mu : \Sigma \to [0,1]$

\+ pogoji

Ali lahko najdemo primer, ko je $\Sigma = P(X)$?

Trivialna mera: izberemo $x \in X$ in za $A \subseteq X$ definiramo

$$\mu(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Mera

Def: Mera na množici X je $\mu : P(X) \to [0,1]$ da:

1) $\mu(\emptyset) = 0$, $\mu(X) = 1$
2) $A \subseteq B \subseteq X \Rightarrow \mu(A) \leq \mu(B)$
3) Če so $\{A_i\}_{i\in\mathbb{N}}$ parome disjunktne $\subseteq X$

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i)$$

4) $\mu(\{x\}) = 0$ za vse $x \in X$

Ali obstaja kakšna mera?

Opomba: X ne more biti končna.

X ne more biti števna: $(x_i)_{i\in\mathbb{N}}$ našteje X

$$1 = \mu(X) = \mu\left(\bigcup_i \{x_i\}\right) = \sum_i \mu(\{x_i\}) = 0$$

Mera μ je dvojiška, če $\mu(A) = 0$ ali $\mu(A) = 1$ za vse $A \subseteq X$.

Obstoj

Ali obstaja dvojiška mera?

Denimo, da je $\mu : P(X) \to \{0,1\}$ dvojiška mera.
Definiramo

$$\mathcal{U}_\mu := \{ A \subseteq X \mid \mu(A) = 1 \}.$$

Lastnosti $\mathcal{U}_\mu$:

1) $\emptyset \notin \mathcal{U}_\mu$, $X \in \mathcal{U}_\mu$

2) $A \subseteq B \subseteq X$ in $A \in \mathcal{U}_\mu \Rightarrow B \in \mathcal{U}_\mu$

3) Če so $\{A_i\}_{i \in \mathbb{N}}$ paroma disjunktne $\subseteq X$

$$\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{U}_\mu \Rightarrow \exists! i \in \mathbb{N}. \; A_i \in \mathcal{U}_\mu$$

4) $\{x\} \notin \mathcal{U}_\mu$ za vse $x \in X$.

- Ali je $\mathcal{U}_x$ zaprt za preseke?

$\mu(A) = 1$ in $\mu(B) = 1 \Longrightarrow \mu(A \cap B) = 1$

5) $A \in \mathcal{U}_\mu$ in $B \in \mathcal{U}_\mu \Longrightarrow A \cap B \in \mathcal{U}_\mu$

6) A, $X \setminus A$: $\forall A \subseteq X .\ A \in \mathcal{U}_\mu \vee X \setminus A \in \mathcal{U}_\mu$.

$\mathcal{U}_\mu$ je <u>ultrafilter</u>, ni glavni, je <u>σ-poln</u>:

(Filter je <u>σ-poln</u> če velja: množica X, filter $F \subseteq P(X)$,

$$\underbrace{\forall \{A_i\}_{i \in \mathbb{N}} \text{ paroma disjunktni}, \bigcup_{i \in \mathbb{N}} A_i = X}_{\text{števna particija } X} \Rightarrow \exists i \in \mathbb{N} .\ A_i \in F.$$

Od ultrafiltra do mere

Denimo, da $\mathcal{U} \subseteq P(X)$ je σ-poln neglavni ultrafilter.

Definiramo

$$\mu(A) = \begin{cases} 1 & A \in \mathcal{U} \\ 0 & A \notin \mathcal{U} \end{cases}$$

Ali je μ mera?

1) $\mu(\emptyset) = 0$, $\mu(X) = 1$

2) monotona ✓

3) σ-aditivnost: paroma disjunktne $\{A_i\}_{i \in \mathbb{N}}$

$$\sum_{i \in \mathbb{N}} \mu(A_i) = \mu\Big(\bigcup_{i \in \mathbb{N}} A_i\Big)$$

$\leq$: če $\exists i.\ A_i \in \mathcal{U} \Rightarrow \bigcup_i A_i \in \mathcal{U}$ zaradi monotonosti $\mathcal{U}$

$\geq$: $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{U} \Rightarrow \exists j.\ A_j \in \mathcal{U}$ zaradi σ-polnosti (premisli!)

4) netrivialna ✓

Polni filtri

Def: Ultrafilter $\mathcal{U}$ na X je κ-poln, $\kappa \in \mathrm{Card}$,
če za vsako particijo $\{A_\alpha\}_{\alpha<\gamma}$ za $\gamma < \kappa$ množice X velja
$\exists \alpha < \gamma .\ A_\alpha \in \mathcal{U}$.

Opomba: σ-poln $\equiv$ $\aleph_1$-poln.