Königova lema

<u>Družina množic</u> je preslikava $A : I \to V$, I je <u>indeksna</u> množica,
pišemo $\{A_i\}_{i \in I}$.

Pozor: $\underbrace{\{A_i \mid i \in I\}}_{\text{slika } A} \neq \underbrace{\{A_i\}_{i\in I}}_{\text{preslikava } A}$, recimo $\{A_i\}_{i \in \omega}$, $A_i = 2$

$\{A_i \mid i \in \omega\} = \{2\}$

$\{2, 2, 2, 2\} = \{2\}$

<u>Koprodukt</u> družine $\coprod_{i \in I} A_i = \sum_{i \in I} A_i := \{(i, x) \in I \times \bigcup_{i \in I} A_i \mid x \in A_i\}$

<u>Produkt</u> $\prod_{i \in I} A_i := \{f : I \to \bigcup_{i \in I} A_i \mid \forall i \in I .\, f(i) \in A_i\}$

<u>Primeri</u>: (1) $I = \{0, 1\}$, $A_0 = B$, $A_1 = C$, $\sum_{i \in I} A_i = B + C$ disjunktna unija

(2) $I = B$, $A_i = C$, $\sum_{i \in I} A_i = B \times C$ $\quad \prod_{i \in I} A_i = {}^{B}C$

Königova lema zares

Lema: Naj bosta $\{\kappa_i\}_{i\in I}$ in $\{\lambda_i\}_{i\in I}$ družini kardinalov ($\kappa : I \to \mathrm{Card}$, $\lambda : I \to \mathrm{Card}$). Če za vsak $i \in I$ velja $\kappa_i < \lambda_i$, potem

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i .$$

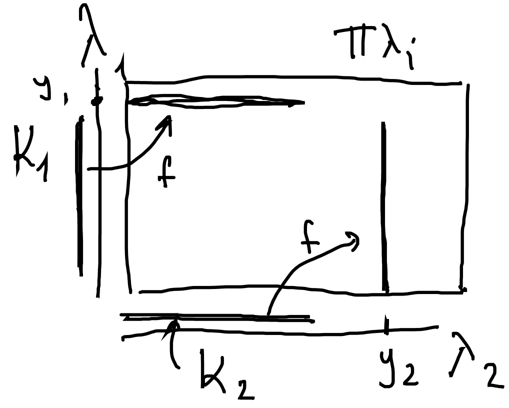
Opomba: Vsota $\underbrace{\sum_i \kappa_i}_{\text{vsota kardinalov}} := \Big| \underbrace{\coprod_i \kappa_i}_{\text{koprodukt množic}} \Big|$ in produkt $\underbrace{\prod_i \lambda_i}_{\text{produkt kardinalov}} := \Big| \underbrace{\prod_i \lambda_i}_{\text{produkt množic}} \Big|$

Opomba:
$$\sum_{i<\omega} 1 = \aleph_0 = \sum_{i<\omega} 2$$

Dokaz

1) Injekcija $\sum_i \kappa_i \to \prod_j \lambda_j$.

Za vsak $i \in I$ je $\kappa_i < \lambda_i$, torej obstaja $g_i : \kappa_i \to \lambda_i$ injektivna in $y_i \in \lambda_i$, ki ni v sliki g_i.



+200

Definiramo $f : \sum_i \kappa_i \to \prod_j \lambda_j$:

$$f(i,x)(j) := \begin{cases} g_i(x) & \text{če } j = i \\ y_j & \text{če } j \neq i. \end{cases}$$

f je injektivna:

$$f(i,x) = f(i',x') \Rightarrow \underbrace{f(i,x)(i)}_{= \, g_i(x)} = \underbrace{f(i',x')(i)}_{\text{če } i \neq i' \;=\; y_i \text{ ni možno}} = g_i(x')$$

torej $i = i'$

$\Rightarrow i = i'$ in $g_i(x) = g_i(x') \Rightarrow x = x'$ ker g injektivna

Dokaz

2) Ni surjekcije $\sum_i \kappa_i \longrightarrow \prod_j \lambda_j$.

Naj bo $h: \sum_i \kappa_i \longrightarrow \prod_j \lambda_j$ poljubna. Dokažemo, da ni surjekcija.

Obravnavamo $i \in I$. Preslikava $\kappa_i \to \lambda_i$ definirana z

$$x \longmapsto h(i,x)(i)$$

ni surjektivna, ker $\kappa_i < \lambda_i$. Torej obstaja $z_i \in \lambda_i$ in ni v sliki te preslikave. −50

$$\forall i \in I\ \exists z \in \lambda_i\ .\ z \notin \text{slika}\,(\overset{\kappa_i \to \lambda_i}{x \mapsto h(i,x)(i)})$$

AC $\exists z : I \to \bigcup_i \lambda_i\ .\ z_i \notin \text{slika}\,(\text{- - - - -})$

Trdimo: preslikava $f \in \prod_j \lambda_j$, definirana z $f(j) = z_j$ ni v sliki h: za vsak $i \in I$, $x \in \kappa_i$

$$h(i,x)(i) \neq z_i = f(i) \implies h(i,x) \neq f$$ ∎ +750

Posledice

Posledica (Cantor): $\mu < 2^\mu$ za $\mu \in \mathrm{Card}$.

Dokaz:

$$\mu = \sum_{i\in\mu} 1 < \prod_{i\in\mu} 2 = 2^\mu.$$

Posledica: Če κ neskončen kardinal, $\kappa < \mathrm{cf}(2^\kappa)$.

Dokaz: Naj bo $f : \kappa \to 2^\kappa$ naraščajoča. Zadostuje $\sup_{\alpha<\kappa} f(\alpha) < 2^\kappa$. Opazimo:

$$\cancel{\sup_{\alpha<\kappa} f(\alpha) \leq} \sum_{\alpha<\kappa} |f(\alpha)| \underset{\text{König}}{<} \prod_{\alpha<\kappa} 2^\kappa = (2^\kappa)^\kappa = 2^{\kappa\cdot\kappa} \underset{\kappa \text{ neskončen}}{=} 2^\kappa.$$

ker

$$|f(\alpha)| \leq f(\alpha) < |f(\alpha)|^+ \leq 2^\kappa \qquad f(\alpha) < 2^\kappa$$

$$|\beta| \leq \beta < |\beta|^+ \qquad \therefore \quad \beta < \mu \Rightarrow |\beta|^+ \leq \mu$$

Posledice

<u>Posledica</u>: za neskončen κ, $\kappa < \kappa^{\mathrm{cf}\,\kappa}$.

Dokaz: Obstaja naraščajoča $f : \mathrm{cf}\,\kappa \to \kappa$, da je

$$\kappa = \sup_{\alpha < \mathrm{cf}\,\kappa} f(\alpha)$$

Naj bo $\kappa_\alpha := |f(\alpha)|$. Tedaj $\kappa_\alpha < \kappa$ za vse $\alpha < \mathrm{cf}\,\kappa$.

$$\kappa \leq \sum_{\alpha < \mathrm{cf}\,\kappa} \kappa_\alpha \underset{\text{König}}{<} \prod_{\alpha < \mathrm{cf}\,\kappa} \kappa = \kappa^{\mathrm{cf}\,\kappa}$$

utemeljitev:

$$\kappa = \left|\sup_{\alpha < \mathrm{cf}\,\kappa} f(\alpha)\right| = \left|\bigcup_{\alpha < \mathrm{cf}\,\kappa} f(\alpha)\right| \leq (\mathrm{cf}\,\kappa) \cdot \sup_{\alpha < \mathrm{cf}\,\kappa} \underbrace{|f(\alpha)|}_{\leq \kappa} \leq (\mathrm{cf}\,\kappa) \cdot \kappa \leq \kappa \cdot \kappa = \kappa$$

$$\left|\bigcup_{\alpha < \mathrm{cf}\,\kappa} f(\alpha)\right| \leq \left|\sum_{\alpha < \mathrm{cf}\,\kappa} f(\alpha)\right|$$

Torej $\left|\sum_{\alpha < \mathrm{cf}\,\kappa} f(\alpha)\right| = \sum_{\alpha < \mathrm{cf}\,\kappa} |f(\alpha)| \geq \kappa$ ✓

Lepo zapisan dokaz:

$$\kappa = |\kappa| = \Big|\sup_{\alpha < \mathrm{cf}\,\kappa} f(\alpha)\Big| = \Big|\bigcup_{\alpha<\mathrm{cf}\,\kappa} f(\alpha)\Big| \overset{(*)}{\leq} \Big|\sum_{\alpha<\mathrm{cf}\,\kappa} f(\alpha)\Big| =$$

$$= \sum_{\alpha<\mathrm{cf}\,\kappa} |f(\alpha)| < \prod_{\alpha<\mathrm{cf}\,\kappa} \kappa = \kappa^{\mathrm{cf}\,\kappa}.$$

König, ker $|f(\alpha)| \leq f(\alpha) < \kappa$,

Utemeljitev $(*)$: $\Big|\bigcup_{i\in I} A_i\Big| \leq \Big|\sum_{i\in I} A_i\Big|$ ker je

$$\sum_{i\in I} A_i \longrightarrow \bigcup_{i\in I} A_i$$

$$(i,x) \longmapsto x$$ surjektivna.