Kofinalnost

<u>Izrek</u>: $\operatorname{cf}(\operatorname{cf}\alpha) = \operatorname{cf}\alpha$.

Dokaz:

$\beta := \operatorname{cf}\alpha$.

Obstaja $f : \beta \to \alpha$ naraščajoča in $\bigcup_{\gamma<\beta} f(\gamma) = \alpha$

$\gamma := \operatorname{cf}\beta$.

Obstaja $g : \gamma \to \beta$ naraščajoča in $\bigcup_{\delta<\gamma} g(\delta) = \beta$.

Funkcija $f \circ g : \gamma \to \alpha$ je naraščajoča, $\bigcup_{\delta<\gamma} f(g(\delta)) = \alpha$.

Torej $\beta \leq \gamma$: $\operatorname{cf}\alpha \leq \operatorname{cf}(\operatorname{cf}\alpha)$.

Vemo že $\operatorname{cf}\beta \leq \beta$; $\operatorname{cf}(\operatorname{cf}\alpha) \leq \operatorname{cf}\alpha$. ∎

Regularni in singularni kardinali

<u>Def</u>: Kardinal κ je <u>regularen</u>, če $\kappa = \mathrm{cf}\,\kappa$.

V nasprotnem primeru je <u>singularen</u>, t.j., $\mathrm{cf}\,\kappa < \kappa$.

Primer:

$\aleph_0$ regularen

$\aleph_\omega$ singularen $\quad \mathrm{cf}\,\aleph_\omega = \omega$, ker $\aleph_\omega = \sup \aleph_0, \aleph_1, \aleph_2, \ldots$

$\aleph_{\alpha+\omega}$ singularen ker $\mathrm{cf}\,\aleph_{\alpha+\omega} = \omega$, ker $\aleph_{\alpha+\omega} = \bigcup_{n<\omega} \aleph_{\alpha+n}$

Najmanjši κ, da je $\aleph_\kappa = \kappa$ je singularen: $\mathrm{cf}\,\aleph_\kappa = \omega$.

Ali obstaja regularen κ, da je $\aleph_\kappa = \kappa$?

Regularni kardinali

Lema: $|\bigcup S| \leq |S| \cdot \sup\{|X| \mid X \in S\}$

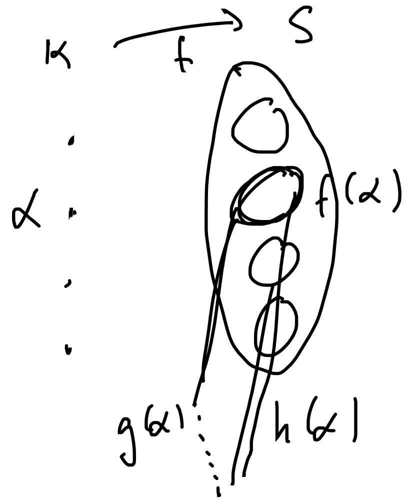
Dokaz: $\kappa := |S|$ $\quad \lambda := \sup\{|X| \mid X \in S\}$. Ideja: surjekcija $\kappa \times \lambda \to \bigcup S$.

Imamo bijekcijo $f : \kappa \to S$.

[Za vsak $\alpha < \kappa$ imamo $f(\alpha) \in S$ in $g(\alpha) \in \mathrm{Ord}$ in

$$h(\alpha) : g(\alpha) \longrightarrow f(\alpha) \text{ bijekcija}]$$

Grrdo.



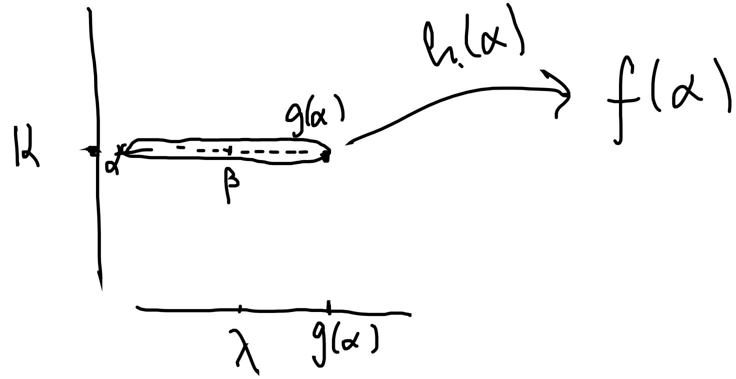
Obstajata g in h, da je za vsak $\alpha < \kappa$

(1) $g(\alpha) \in \mathrm{Card}$ in $g(\alpha) \leq \lambda$

(2) $h(\alpha) : g(\alpha) \longrightarrow f(\alpha)$ bijekcija

Uporabili smo AC!

$$\boxed{g(\alpha) := |f(\alpha)|}$$



- Imamo: $\cancel{T = \{(\alpha,\beta) \mid \alpha<\kappa \text{ in } \beta<g(\alpha)\} \subseteq \kappa\times\lambda}$

<u>Izboljšava:</u> imamo $f : \kappa \to \bigcup S$ bijekcije

Definiramo $g(\alpha) := |f(\alpha)|$

$h(\alpha) : g(\alpha) \to f(\alpha)$ bijekcija

$$T = \{(\alpha,\beta) \mid \alpha<\kappa \text{ in } \beta<g(\alpha)\} \subseteq \kappa\times\lambda$$

$T \to \bigcup S$

$(\alpha,\beta) \mapsto h(\alpha)(\beta)$ je surjektivna: $x \in \bigcup S \overset{f \text{ surjekcija}}{\Rightarrow} \exists \alpha<\kappa .\ x \in f(\alpha).$

$\underset{h(\alpha) \text{ surjektivna}}{\Rightarrow} \exists \beta < g(\alpha).$

$x = h(\alpha)(\beta)$

$$|\bigcup S| \le |T| \le |\kappa\times\lambda| = \kappa\cdot\lambda.$$

$\blacksquare$

- Lema je posplošitev: števna unija števnih množic je števna.

S števna, $\forall x \in S.\ x$ števna

$|S| \leq \aleph_0$ $\qquad |x| \leq \aleph_0$

$$|\bigcup S| \leq |S| \cdot \sup\{|x| \mid x \in S\}$$

$$\leq \aleph_0 \cdot \aleph_0 = \aleph_0.$$

premisli pred spanjem

Ali je $\aleph_\beta$ limitni ordinal
Ali so neskončni kardinali limitni ordinali

<u>Posledica</u>: $\aleph_{\alpha+1}$ je regularen.

<u>Dokaz</u>: Denimo $\operatorname{cf} \aleph_{\alpha+1} < \aleph_{\alpha+1}$. Potem bi imeli

$$f : \beta \to \aleph_{\alpha+1}, \quad \beta < \aleph_{\alpha+1} \quad \text{in} \quad \bigcup_{\alpha < \beta} f(\alpha) = \aleph_{\alpha+1}.$$

$$\aleph_{\alpha+1} = |\aleph_{\alpha+1}| = \Big|\bigcup_{\alpha<\beta} f(\alpha)\Big| \underset{\text{lema}}{\leq} |\beta| \cdot \sup\{|f(\alpha)| \mid \alpha < \beta\}$$

$$\leq \aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha \qquad \text{protislovje.} \ \blacksquare$$