Normalne funkcije

Def: $f: \mathrm{Ord} \to \mathrm{Ord}$ je normalna, če

(1) naraščajoča: $\alpha < \beta \Rightarrow f(\alpha) < f(\beta)$

(2) zvezna:

$$f(\beta) = \bigcup_{\alpha < \beta} f(\alpha) \quad \text{za vse limitne } \beta$$

Primeri:

1) Seštevanje, naj bo $\alpha \in \mathrm{Ord}$.

Ali je $f(\beta) = \alpha + \beta$ normalna?

- naraščajoča? $\beta < \gamma \Rightarrow \alpha + \beta < \alpha + \gamma$ ✓
- zvezna: β limitni

$$\bigcup_{\gamma < \beta} f(\gamma) = \bigcup_{\gamma < \beta} \alpha + \gamma = \alpha + \beta = f(\beta)$$

Naloga (doma)

Normalne funkcije

$f(\beta) = \beta + \alpha$ normalna? $\alpha = 0$ ✓

$\omega > \alpha > 0$?

$\alpha \geq \omega$ ni naraščajoča

$\alpha = 1$: $f(\beta) = \beta + 1$

- naraščajoča $\beta < \gamma \Rightarrow \beta + 1 < \gamma + 1$? ✓
- zvezna ? β limitni

$$\bigcup_{\gamma < \beta} \gamma + 1 = \beta \neq \beta + 1$$ ni zvezna!

Primer: $\aleph : \mathrm{Ord} \to \mathrm{Card} \subseteq \mathrm{Ord}$.

- zvezna? β limitni,

$$\bigcup_{\alpha < \beta} \aleph_\alpha = \aleph_\beta$$ po definiciji

Alef

$$\alpha < \beta \overset{?}{\Longrightarrow} \aleph_\alpha < \aleph_\beta \ ?$$

Najprej: $\alpha \leq \beta \Longrightarrow \aleph_\alpha \leq \aleph_\beta$? $\qquad \alpha = \beta \Rightarrow \aleph_\alpha \leq \aleph_\beta$ ✓

$\alpha < \beta \Rightarrow \aleph_\alpha \leq \aleph_\beta$? Indukcija po β:

- če β limitni: $\aleph_\alpha \leq \bigcup_{\gamma<\beta} \aleph_\gamma = \aleph_\beta$
- če $\beta = \gamma + 1$: $\quad \alpha < \gamma + 1$

$$\alpha \leq \gamma \underset{\mathrm{IH}}{\Longrightarrow} \aleph_\alpha \leq \aleph_\gamma \overset{\text{def } \aleph}{<} \aleph_{\gamma+1} = \aleph_\beta$$

Dokažimo: $\alpha < \beta \Rightarrow \aleph_\alpha < \aleph_\beta$

- če $\beta = \gamma + 1$; $\quad \alpha < \gamma + 1 \Rightarrow \alpha \leq \gamma$

$$\aleph_\alpha \leq \aleph_\gamma < \aleph_{\gamma+1} = \aleph_\beta .$$

- če β limitni: $\gamma < \beta \Longrightarrow \gamma < \gamma + 1 < \beta \Longrightarrow$ $\qquad [\alpha \leq \gamma]$

$$\aleph_\gamma < \aleph_{\gamma+1} \leq \aleph_\beta$$

Lemi

<u>Lema</u>: f normalna $\Rightarrow$ $\alpha \leq f(\alpha)$ za vse $\alpha \in \mathrm{Ord}$.

Dokaz: Denimo, ~~da~~ ~~obstaja~~ protiprimer.
Naj bo α najmanjši tak, da $f(\alpha) < \alpha$.
Tedaj je $f(\alpha)$ še manjši protiprimer, saj
$f(\alpha) < \alpha \Rightarrow f(f(\alpha)) < f(\alpha)$ ker f narašča. $\blacksquare$

<u>Lema</u>: Če f normalna in $S \subseteq \mathrm{Ord}$ podmnožica,

$$f(\cup S) = \bigcup_{\alpha \in S} f(\alpha).$$

<u>Dokaz</u>:

$$\alpha \leq \bigcup_{\beta \in S} \beta \quad \text{za } \alpha \in S$$

$$f(\alpha) \leq f\Big(\bigcup_{\beta \in S} \beta\Big) \quad \text{za } \alpha \in S$$ torej $f(\cup S)$ zgornja meja $\{f(\alpha) \mid \alpha \in S\}$

$$\bigcup_{\alpha \in S} f(\alpha) \leq f(\cup S) \quad \checkmark$$

- $f(\bigcup S) \subseteq \bigcup_{\alpha \in S} f(\alpha)$ dokažimo:

Naj bo $\delta = \bigcup S$.

1) $\delta = 0$: premisli doma

2) $\delta = \gamma + 1$: $\bigcup S = \gamma + 1 \Rightarrow \delta = \gamma + 1 \in S$

$$f(\delta) \subseteq \bigcup_{\alpha \in S} f(\alpha) \quad \text{ker } \delta \in S.$$

3) δ limitni:

$$f(\delta) = f\Big(\bigcup_{\alpha < \delta} \alpha\Big) \overset{\text{zvezna}}{=} \bigcup_{\alpha < \delta} f(\alpha) \subseteq \bigcup_{\beta \in S} f(\beta)$$

Ker za vsak $\alpha < \delta = \bigcup S$, obstaja $\beta \in S$, $\alpha < \beta$, ker f narašča $f(\alpha) < f(\beta)$.

$\blacksquare$

Negibne točke normalne funkcije

Izrek: Normalna funkcija f ima negibno točko nad vsakim $\alpha \in \mathrm{Ord}$. Torej je $\{\beta \in \mathrm{Ord} \mid f(\beta) = \beta\}$ pravi razred.

Dokaz: Naj bo $\alpha \in \mathrm{Ord}$. Definiramo

$$\beta_0 = \alpha$$
$$\beta_{n+1} = f(\beta_n) \qquad \text{za } n \in \omega$$

$$\beta_0 \underset{\substack{\uparrow \\ \text{ena lema}}}{\leq} \overbrace{f(\beta_0)}^{\beta_1} \leq \underbrace{f(f(\beta_0))}_{\beta_2} \leq \cdots$$

$$f\Big(\bigcup_{n\in\omega} \beta_n\Big) \underset{\substack{\uparrow \\ \text{lema}}}{=} \bigcup_{n\in\omega} f(\beta_n) = \bigcup_{n\in\omega} \beta_{n+1} = \bigcup_{n\in\omega} \beta_n$$

Torej je $\beta = \bigcup_{n\in\omega} \beta_n$ negibna za f. Očitno $\alpha \leq \beta$. ∎

Doma: Ta β je najmanjša negibna točka f, ki je $\geq \alpha$.

Primeri

- Funkcija $f(\alpha) = 1+\alpha$ je normalna.
  Negibne točke: ω, $\omega+1$, vsi neskončni ordinali

- Funkcija $f(\alpha) = \beta \cdot \alpha$ za izbrani $\beta \in Ord$, $\beta > 0$ (premisli ali normalna)
  Negibne točke: $\beta^{\omega} = \beta^{1+\omega} = \beta^{1} \cdot \beta^{\omega} = \beta \cdot \beta^{\omega}$

  $1, \beta \cdot 1, \beta \cdot \beta \cdot 1, \beta \cdot \beta \cdot \beta \cdot 1,$
  $\beta^{0}, \beta^{1}, \beta^{2}, \ldots, \xrightarrow[\sup]{} \beta^{\omega}$

- Funkcija $f(\alpha) = \omega^{\alpha}$ je normalna
  $0, \omega^{0}, \omega^{\omega^{0}}, \omega^{\omega^{\omega^{0}}}, \ldots \xrightarrow[\sup]{} \varepsilon_0$ $\qquad \omega^{\varepsilon_0} = \varepsilon_0$

Primer $\aleph$ je normalna. Torej obstaja tako $\kappa \in \mathrm{Card}$, da je

$$\aleph_\kappa = \kappa .$$

κ je supremum $0, \aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \ldots$

$$\kappa = \aleph_{\aleph_{\aleph_{\aleph_{\aleph_{\cdot_{\cdot_{\cdot}}}}}}}$$

$\aleph_0 \quad \aleph_1 \quad \aleph_{\aleph_0}$

$$\aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots$$

$$0 \quad 1 \quad 2 \quad \cdots \quad \omega \quad \omega+1 \quad \omega+2 \quad \cdots$$

Aritmetični zakoni

Za $\kappa, \lambda, \mu \in \mathrm{Card}$ velja:

$$\kappa + \lambda = \lambda + \kappa \qquad \kappa \cdot \lambda = \lambda \cdot \kappa$$

$$(\kappa + \lambda) + \mu = \kappa + (\lambda + \mu) \qquad (\kappa \cdot \lambda) \cdot \mu = \kappa \cdot (\lambda \cdot \mu)$$

$$\kappa + 0 = \kappa \qquad \kappa \cdot 1 = \kappa$$

$$(\kappa + \lambda) \cdot \mu = \kappa \cdot \mu + \lambda \cdot \mu \quad ? \quad [\text{doma}]$$

$\kappa^0 = 1$ (posebej $0^0 = 1$)

$1^\kappa = 1$

$\kappa^{\lambda + \mu} = \kappa^\lambda \cdot \kappa^\mu \qquad A^{B+C} \cong A^B \times A^C$

$\kappa^{\lambda \cdot \mu} = (\kappa^\lambda)^\mu$

Aritmetika neskončnih kardinalov

Izrek: κ neskončen kardinal $\Rightarrow$ $\kappa \cdot \kappa = \kappa$.

Dokaz: Vaje.

Izrek: Naj bosta κ, λ neskončna kardinala. Tedaj

$$\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda).$$

Dokaz: Obravnavamo le $\kappa \leq \lambda$:

- $\max(\kappa,\lambda) = \lambda \leq \kappa \cdot \lambda \leq \lambda \cdot \lambda = \lambda = \max(\kappa,\lambda)$

  doma $\quad [\kappa \leq \lambda \Rightarrow \kappa \cdot \mu \leq \lambda \cdot \mu]$

- $\max(\kappa,\lambda) = \lambda \leq \kappa + \lambda \leq \lambda + \lambda = 2 \cdot \lambda \leq \lambda \cdot \lambda = \lambda = \max(\kappa,\lambda)$.

$\blacksquare$

Potenciranje (κ, λ neskončna kardinala)

<u>Posledica</u> Če je $\kappa \leq \lambda$, potem $\kappa^\lambda = 2^\lambda$.

Dokaz:
$$2^\lambda \underset{\substack{\uparrow \\ \text{doma}}}{\leq} \kappa^\lambda \leq (2^\kappa)^\lambda = 2^{\kappa \cdot \lambda} = 2^\lambda.$$

<u>Posledica</u>: Če $\kappa > \lambda$, potem $\kappa^\lambda \leq 2^\kappa$.

Dokaz:
$$\kappa^\lambda \leq (2^\kappa)^\lambda = 2^{\kappa \cdot \lambda} = 2^\kappa.$$

Kofinalnost

Definicija: Kofinalnost limitnega ordinala α je

$\operatorname{cf}\alpha :=$ najmanjši limitni $\beta \in \mathrm{Ord}$, da obstaja naraščajoča $f : \beta \to \alpha$, da je $\alpha = \bigcup_{\gamma<\beta} f(\gamma)$.

$$:= \min\{\beta \in \mathrm{Ord} \mid \beta \text{ limitni in } \exists f : \beta \to \alpha,\ f \text{ naraščajoča} \wedge \alpha = \bigcup_{\gamma<\beta} f(\gamma)\}.$$

Primeri: $\operatorname{cf} 0 = 0$

$\operatorname{cf}\omega = \omega$ ker $\underbrace{f : n \to \omega}_{\substack{\text{končno zap.}\\ \text{končnih ordinalov}}}$, $n < \omega \Rightarrow \bigcup_{k<n} f(k) < \omega$, torej $\operatorname{cf}\omega \geq \omega$.

$\operatorname{cf}\alpha \leq \alpha$ ker $\alpha = \bigcup_{\beta<\alpha}\beta$, torej $f = \mathrm{id} : \alpha \to \alpha$

Primeri

$\mathrm{cf}(\omega+\omega) = \omega$ $\qquad \omega+\omega = \sup\{\omega, \omega+1, \omega+2, \omega+3, \omega+4, \dots\}$

$\mathrm{cf}(\omega^\omega) = \omega$ $\qquad \omega^\omega = \sup\{\omega^1, \omega^2, \omega^3, \omega^4, \dots\}$

$\mathrm{cf}(\varepsilon_0) = \omega$ $\qquad \varepsilon_0 = \sup\{\omega^0, \omega^{\omega^0}, \omega^{\omega^{\omega^0}}, \dots\}$

$\mathrm{cf}(\aleph_\kappa)$ za $\kappa = \aleph_\kappa$ najmanjša negibna točka $\aleph$?

$\quad \|$

$\quad \omega$ $\qquad \aleph_\kappa = \sup\{0, \aleph_0, \aleph_{\aleph_0}, \dots\}$

Ali so kje kakšni ordinali, ki imajo $\mathrm{cf} > \omega$?