Ordinalna aritmetika

$$\alpha + 0 = \alpha$$

$$\alpha + \beta^{+} = (\alpha + \beta)^{+}$$

$$\alpha + \beta = \sup_{\gamma < \beta} \alpha + \gamma \qquad \beta \text{ limitni}$$

$$\alpha^{0} = 1$$

$$\alpha^{\beta^{+}} = \alpha^{\beta} \cdot \alpha$$

$$\alpha^{\beta} = \sup_{\gamma < \beta} \alpha^{\gamma}$$

$$\alpha \cdot 0 = 0$$

$$\alpha \cdot \beta^{+} = \alpha \cdot \beta + \alpha$$

$$\alpha \cdot \beta = \sup_{\gamma < \beta} \alpha \cdot \gamma \qquad \beta \text{ limitni}$$

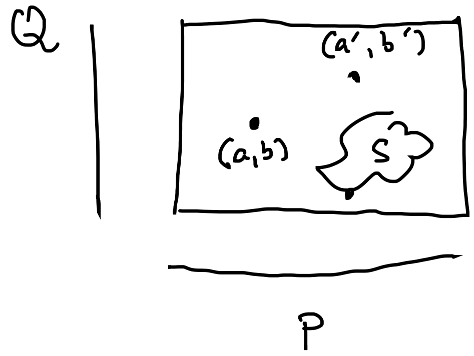
$$\boxed{\begin{array}{l} \beta^{+} = \beta + 1 \qquad \alpha^{\beta + \gamma} = \alpha^{\beta} \cdot \alpha^{\gamma} \\ \alpha^{\beta + 1} = \alpha^{\beta} \cdot \alpha^{1} = \alpha^{\beta} \cdot \alpha \end{array}}$$

$$\alpha \cdot 2 = \alpha \cdot 1 + \alpha = \alpha + \alpha$$

$$2 \cdot \alpha$$

Množenje

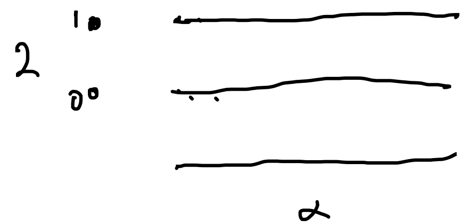
$$(P, <_P) \cdot (Q, <_Q)$$



leksikografska ureditev

$$(a,b) <_{P \times Q} (a',b') \iff$$

$$b <_Q b' \lor (b = b' \land a <_P a')$$



$$\emptyset \neq S \subseteq P \times Q$$

$$b := \min\{c \in Q \mid \exists a \in P .\ (a,c) \in S\}$$

$$a := \min\{d \in P \mid (d,b) \in S\}$$

$$\min S = (a,b)$$

Računamo

$$\omega \cdot 2 = \omega + \omega$$

$$2 \cdot \omega = \sup_{n<\omega} 2 \cdot n = \sup\{0, 2, 4, 6, 8, \dots\} = \omega$$

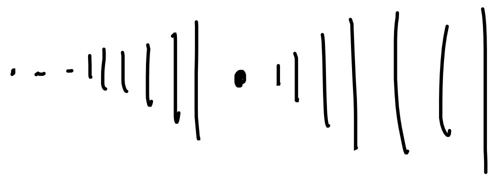
$$2^{\omega} = \sup_{n<\omega} 2^{n} = \sup\{1, 2, 4, 8, 16, \dots\} = \omega$$

α^{β} na sliki?

Komutativnost: $\underbrace{1+\omega}_{=\omega} \neq \underbrace{\omega+1}_{=\omega^{+}}$ $\qquad \underbrace{2\cdot\omega}_{=\omega} \neq \underbrace{\omega\cdot 2}_{=\omega+\omega}$

Asociativnost?

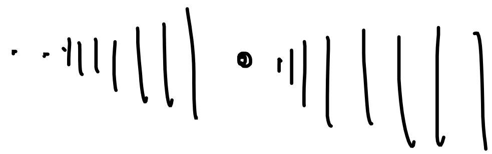
$$\omega+\omega = (\omega+1)+\omega = \cancel{\omega\omega}$$



$$\omega^{+} + \omega = \omega + \omega$$

$$0 + \omega = 1 + \omega$$

$$\omega+\omega = \omega + \underbrace{(1+\omega)}_{=\omega}$$



Asociativnost +

Izrek: $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$.

Dokaz: Indukcija po γ:

1) $(\alpha+\beta)+0 = \alpha+\beta$

$\alpha+(\beta+0) = \alpha+\beta$

2) $(\alpha+\beta)+\delta^+ = ((\alpha+\beta)+\delta)^+ \overset{\text{I.H.}}{=} (\alpha+(\beta+\delta))^+ = \alpha+(\beta+\delta)^+ = \alpha+(\beta+\delta^+)$

3) γ limitni, $\gamma\neq 0$:

$(\alpha+\beta)+\gamma = \lim\limits_{\delta<\gamma}(\alpha+\beta)+\delta \overset{\text{I.H}}{=} \sup\limits_{\delta<\gamma}\alpha+(\beta+\delta)$

Če definiramo $f(\xi) = \alpha+\xi$

$\alpha+(\beta+\gamma) = \alpha+(\sup\limits_{\delta<\gamma}\beta+\delta)$

Zakaj $\sup\limits_{\delta<\gamma} f(\beta+\delta) \overset{?}{=} f(\sup\limits_{\delta<\gamma}\beta+\delta)$

OK, če je f "zvezna".

Seštevanje in supremumi Izrek: Za vsak $\alpha \in \mathrm{Ord}$:

Naj bo $S \subseteq \mathrm{Ord}$, množica. Tedaj je $\alpha + \sup S = \sup\{\alpha + \beta \mid \beta \in S\}$.

Dokaz: Indukcija po α.

1) $\alpha = 0$: $0 + \sup S = \sup S$ ker $0 + \beta = \beta$ (indukcija na β)

$\sup\{0 + \beta \mid \beta \in S\} = \sup\{\beta \mid \beta \in S\} = \sup S$

2) $\alpha = \delta^+$: $\delta^+ + \sup S$

$\sup\{\delta^+ + \beta \mid \beta \in S\}$

ABORT.

$\sup M \leq x$

$x \leq \sup M$

Seštevanje in supremumi

$$\alpha + \sup S = \sup\{\alpha+\beta \mid \beta \in S\}$$

$\boxed{\geq}$ $\sup\{\alpha+\beta \mid \beta\in S\} \leq \alpha + \sup S$ če je $\forall \beta \in S.\ \alpha+\beta \leq \alpha + \sup S$.

Vzemimo $\beta \in S$. Tedaj je $\beta \leq \sup S$.

Torej $\alpha+\beta \leq \alpha + \sup S$. ZAKAJ?

$\boxed{\text{Lema: } \beta \leq \gamma \Rightarrow \alpha+\beta \leq \alpha+\gamma.}$

$\boxed{\leq}$ $\alpha + \sup S \leq \sup\{\alpha+\beta \mid \beta \in S\}$?