HOMOTOPY (TYPE) THEORY

19.2.2019

1. HOMOTOPY THEORY

MOTIVATION (FOR HT)

What does maths deal with?
Physics: GEOMETRIC OBJECTS for example manifolds $M \subset \mathbb{R}^n$
Motions in M: $\gamma : [a,b] \to M$
- we need to differentiate

Distances are important in geometry.
How to distinguish between M and M'?
Relaxation N°1: View M and M' just as smooth manifolds

it is a different manifold, but the same as topological space

N°2: Relax to topological spaces

Note: topological spaces (with continuous maps) is a category.

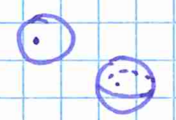
To distinguish spaces we use invariants: if the objects don't have the same invariant, they are different

Like functorial invariants: an invariant on $\mathcal{C}$ with values in $\mathcal{D}$ is a functor from $\mathcal{C}$ to $\mathcal{D}$.
(Examples of non-functorial invariants: Euler characteristics, dimension, ...)

Notorious invariants on topological spaces

- HOMOTOPY GROUPS (they mesure HOLES in a topological space)

$\mathbb{R}^2 \setminus \{0\}$
$\mathbb{R}^3 \setminus \{0\}$



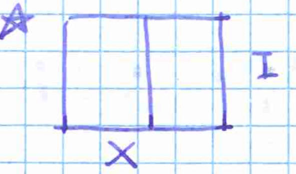
$\Pi_n : \underline{Top^*} \to \underline{Grp}$
↳ topological spaces with distinguished point

- (co) HOMOLOGY GROUPS

all of the above factor through the homotopy category

Def. The homotopy category of topological spaces:
Recall, that two maps $f, g : X \to Y$ are homotopic if there $\exists$
$h : X \times [0,1] \to Y$ with $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all x.

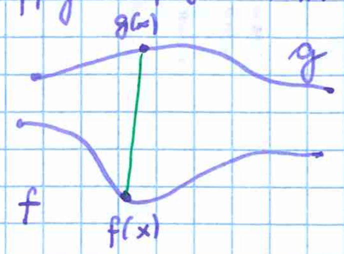
$t \rightsquigarrow h_t : X \to Y$, $x \mapsto h(x,t)$ parametrized deformation of f to g, a continuous deformation, because it is cont. on the product

☆ 

$h\big|_{\{x\}\times[0,1]}$ is a path in Y from $f(x)$ to $g(x)$.

h is a globally coherent choice of paths $f(x) \rightsquigarrow g(x)$

Example: $f, g: [0,1] \longrightarrow \mathbb{R}^2$ paths in plane



$(1-t)\, f(x) + t\, g(x)$

(domain can be any convex set in a top. space)
we just connect with a straight line

Example. $\iota: S^1 \longrightarrow \mathbb{R}^2 \setminus \{0\}$ (inclusion)
$c_{(1,0)}: S^1 \longrightarrow \{(1,0)\} \hookrightarrow \mathbb{R}^2\setminus\{0\}$ (constant path)
are not homotopic

HoTop: homotopy category
Objects: topological spaces
Morphisms $X \to Y$: equivalence classes $[f]$ of continuous maps f
↳ homotopic is obviously an equivalence relation
easy observation — refl: constantly f, sym: $h(x, 1-t)$, trans.: go twice as fast

compositions: $[g]\circ[f] = [g\circ f]$
(have to check that if $f \simeq f'$ and $g \simeq g'$, then $g\circ f \simeq g'\circ f'$
so it is well defined)

Identities: $[id: X \longrightarrow X]$

X and Y are isomorphic in HoTop if there $\exists$ cont. maps
$f: X \longrightarrow Y$, $g: Y \to X$ with $g\circ f \simeq id_X$ and $f\circ g \simeq id_Y$

Because the interesting invariants factor through HoTop this is the category of interest. We will only relax this notions at the end of the first part of the course

Note: We can have forgetful functors geom. objects → smooth manifolds → Top. spaces → HoTop
if we differentiate objects in HoTop, then we differentiate in categories with more structure.

Def. Relative homotopy: Suppose for $A \subset X$ we have $f(x) = g(x)$ for $x \in A$. Then $f \simeq_A g$ if a homotopy $h: X\times[0,1] \longrightarrow Y$ exists that is stationary on A ($h(x,t) = f(x)$ for all t and all $x \in A$)

Important case: Top^*: objects (X, x_0), where $x_0 \in X$, morphisms have $f(x_0) = y_0$ (base point of X goes to base point of Y)
Can form $HoTop^*$ by taking homotopies rel. basepoint.

Remark. Can generalize to category under A with
$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X \arrow[rr] & & Y \end{tikzcd}$$

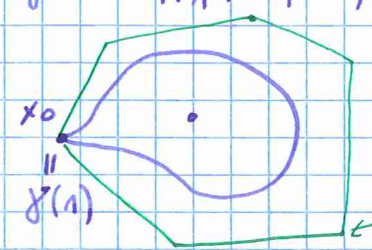
The fundamental group

$\pi_1: Top^* \longrightarrow Grp$

Given (X, x_0), define the loop space $\Omega(X, x_0) = \mathcal{C}((S^1, 1), (X, x_0))$
continuous maps — complex one

when we want to say "canonical basepoint" we write $\mathcal{C}((S^1, *), (X, *))$

Define $\pi_1(X, x_0) = \Omega(X, x_0)/\simeq_*$ ← homotopy rel basepoint



$x_0 = \gamma(1)$

$\gamma: S^1 \longrightarrow (\mathbb{R}^2 \setminus \{0\}, x_0)$

← an entire interval of time can be spent on this point

↑ these are homotopic rel x_0
they are not homotopic to the constant path in x_0

Opomba: The topology is compact open topology on loops

Group structure

$[\gamma]\cdot[\delta] = [\gamma \star \delta]$ — concatenation

$$\gamma \star \delta(z) = \begin{cases} \gamma(z^2), & z = e^{2\pi i t} \text{ for } t \leq \frac{1}{2} \\ \delta(z^2), & z = e^{2\pi i t} \text{ for } t \geq \frac{1}{2} \end{cases}$$

⌒ traverses γ and ⌣ traverses δ

Neutral element $[const_*]$

Inverse: $\bar{\gamma}(z) = \gamma(\bar{z})$ we reverse the direction of traversing

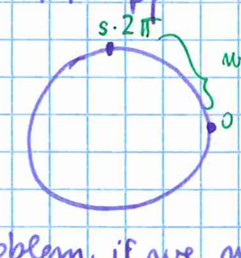
$[\gamma]^{-1} := [\bar{\gamma}]$

proof that it is an inverse:
Have to show $[\gamma]\cdot[\gamma^{-1}] = [\gamma \star \bar{\gamma}] = [const_*]$

$h(z, s) = \left(\gamma\big|_{[0,s]\cdot 2\pi\text{-arc}}\right) \star \left(\bar{\gamma}_{[0,s]\cdot 2\pi\text{-arc}}\right)$

homotopy between $\gamma \star \bar{\gamma}$ and $const_*$

↑ these concatenations are well defined!!
(the end point of the first part is the starting point of the second part)

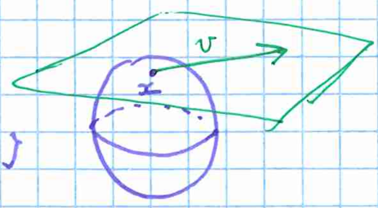


$s\cdot 2\pi$, 0 — we restrict γ to this portion of the entire 2π arc

op. Problem, if we want smooth things and we do concatenation, we might not get a smooth thing ⌃ ⇝ we need to do something, spend some time in the point in C^∞ to make this work → Physicists do this.

BUNDLES (MOTIVATION FOR FIBRATIONS)

Geometric tangent bundle:



we mean the linear plane through the origin, not the affine one

M smooth Riemannian manifold $\subset \mathbb{R}^N$

$TM = \{(x,v) \mid x \in M,\ v \text{ tangent to } M \text{ at } x\}$

$(TM)_x = \{v \mid v \text{ tangent to } M \text{ at } x\}$

If M is m-dimensional, then $(TM)_x \cong \mathbb{R}^m$

(Motivation, motion $\gamma: (-\varepsilon,\varepsilon) \to M$ smooth with $\gamma(0) = x$, $\dot\gamma(0) = v$)

velocity is in tangent space

If we take all velocity vectors, we get the entire tangent space

geometric, not intrinsic, geometric means it is in $\mathbb{R}^N$

If $f: M \to M'$ smooth map between manifolds, then get $Tf: TM \to TM'$ defined by $Tf(x,v) = (f(x), Df(x)v)$

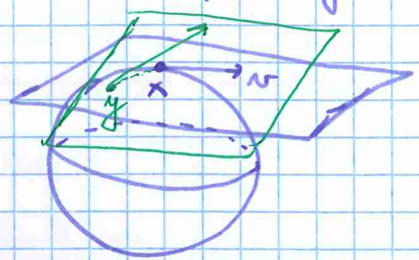
not too important for the rest of the talk

tangent bundles are smooth of a degree one less than M

derivative of f in x acting on v

Tangent planes move/vary when we change x. What about locally?

For y close to x, TM_y is "close" to TM_x



when y is close to x we can take a projection as the rain is falling to get an isomorphism $TM_y \cong TM_x$

- Actually for r small enough

the first component is identity, the second depends on M and neighbourhood

$$TM_{B(x,r)} = \{(y,v) \mid (y,v) \in TM,\ y \in B(x,r)\} \cong B(x,r) \times \mathbb{R}^m$$

(Note for each y TM_y maps ISOMORPHICALLY onto $\mathbb{R}^m$)

in the sense of vector spaces/linearity

In applications, want to associate to $x \in M$ other spaces (vector spaces or more general)

GENERAL BUNDLE

Digression:

Suppose B is a space (an object in the Topological category) or any category $\mathcal{C}$

We can form category of objects OVER B (slice category)

$\mathcal{C}_B$ with objects $\begin{array}{c} X \\ \downarrow p \\ B \end{array}$ and morphisms

commutative diagrams

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[r, "g"] \arrow[d, "q"] & Z \arrow[dl, "r"] \\ & B & \end{tikzcd}$$

composition

and the identity is obvious

(this is obviously a category and it is dual to "under" category)

Def. A trivial object over B is the "product" $\begin{array}{c} B\times F \\ \downarrow pr \\ B \end{array}$ for some F (and all isomorphic objects) we treat isomorphic as equal

(in a category with finite products)

Note that $\mathrm{Hom}_{\mathcal{C}_B}\left(\begin{array}{c} X \\ \downarrow \\ B \end{array}, \begin{array}{c} B\times F \\ \downarrow \\ B \end{array}\right) \cong \mathrm{Hom}_{\mathcal{C}}\left(\mathcal{U}\left(\begin{array}{c} X \\ \downarrow \\ B \end{array}\right), F\right) = \mathrm{Hom}_{\mathcal{C}}(X, F)$

adjoint pair $\mathcal{U}$ and F

forgetful functor

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "p"'] & & B\times F \arrow[dl, "pr"] \\ & B & \end{tikzcd}$$

(it is defined on p and B, only $X \to F$ is the free part)

this is a way to make bundles from F

Suppose we have

- a map $p: E \to B$
  - p projection, E espace, B base from french
- a space F
- an effective action $G \times F \to F$
  - Adjoint $G \to \mathrm{Homeo}(F)$ is injective
  - if an element acts as identity on G it is an identity

the above constitute a locally trivial bundle over B with general fibre F and structure group G if there is an open covering $\{U_\alpha \mid \alpha \in A\}$ of B such that:

(1) $E|_{U_\alpha} = p^{-1}(U_\alpha) \cong U_\alpha \times F$

as spaces over U_α

$$\begin{tikzcd} E|_{U_\alpha} \arrow[rr, "\varphi_\alpha"] \arrow[dr, "p"'] & & U_\alpha \times F \arrow[dl, "pr"] \\ & U_\alpha & \end{tikzcd}$$

restriction

local trivialisation (bundle chart)

(2) Suppose $U_\alpha \cap U_\beta \neq \emptyset$

Then $\varphi_\beta \circ \varphi_\alpha^{-1} : U_\alpha \cap U_\beta \times F \longrightarrow U_\alpha \cap U_\beta \times F$

should be of the form $(x,v) \longmapsto (x, \varphi_{\beta\alpha}(x)\cdot v)$

acting on v

(action is global)

where $\phi_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ is a continuous map. ↙ what is the topology on G?

Why we do this: we define a bundle by glueing things together.

Note: $(\phi_\alpha \circ \phi_\beta^{-1}) \circ (\phi_\beta \circ \phi_\alpha^{-1}) = id$

this forces $\phi_{\beta\alpha}$ and $\phi_{\alpha\beta}$ to compose to id

For $F = \mathbb{R}^m$, $G = GL(m, \mathbb{R})$ with cannonical action $GL(m,\mathbb{R}) \times \mathbb{R}^m \to \mathbb{R}^m$ get the concept of rank m vector bundle.

26.2.2019

Recall

Let $p : E \to B$ be a map and $G \times F \to F$ an effective action of the topological group G on the space F.

Then p is a locally trivial fibre bundle with general fibre F and structure group G if there exists an open covering $\{U_\alpha \mid \alpha \in A\}$ of B s.t.

(1) $p^{-1}(U_\alpha) \underset{def}{=} E|_{U_\alpha} \xrightarrow[\cong]{\Phi_\alpha} U_\alpha \times F$ (local triviality)

$$\begin{tikzcd} E|_{U_\alpha} \arrow[rr, "\Phi_\alpha", "\cong"'] \arrow[dr, "p"'] & & U_\alpha \times F \arrow[dl, "pr"] \\ & U_\alpha & \end{tikzcd}$$

(2) $\forall \alpha, \beta \ \exists \phi_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ s.t. $\Phi_\beta \circ \Phi_\alpha^{-1} : U_\alpha \cap U_\beta \times F \to U_\alpha \cap U_\beta \times F$ has the form

$(x, v) \mapsto (x, \phi_{\beta\alpha}(x) \cdot v)$

If we omit (2) we just have locally trivial fibre bundle

the maps $\phi_{\beta\alpha}$ are called transition functions.

Actions $G \times F \to F$ are mostly in bijective correspondence with $G \to Homeo(F)$ (the largest group that makes sense)

Example. $F = \mathbb{R}^n$ and $G = GL(n, \mathbb{R})$ get the concept of vector bundle of rank n.

Note, that the fibres have "compatible" vector space structures

Let $x \in B$ and $E_x = p^{-1}(x)$. (one fibre over x)

given $u_1, u_2 \in E_x$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, write

$\Phi_\alpha(u_j) = (x, v_j)$;

set $\lambda_1 u_1 + \lambda_2 u_2 := \Phi_\alpha^{-1}(x, \lambda_1 v_1 + \lambda_2 v_2)$

suppose also $x \in U_\beta$; then $\Phi_\beta(u_j) = (x, w_j)$ and would want $\Phi_\beta^{-1}(x, \lambda_1 w_1 + \lambda_2 w_2)$

Now we have $\lambda_1 u_1 + \lambda_2 u_2 = \Phi_\alpha^{-1}(x, \lambda_1 v_1 + \lambda_2 v_2) = \Phi_\beta^{-1} \circ \Phi_\beta \circ \Phi_\alpha^{-1}(x, \lambda_1 v_1 + \lambda_2 v_2) =$
$= \Phi_\beta^{-1}(x, \lambda_1 \underbrace{\phi_{\beta\alpha}(x) v_1}_{w_1} + \lambda_2 \underbrace{\phi_{\beta\alpha}(x) v_2}_{w_2})$

$\Phi_\beta(u_j) = \Phi_\beta \circ \Phi_\alpha^{-1} \circ \Phi_\alpha(u_j) = (x, \underbrace{\phi_{\beta\alpha}(x) \cdot v_j}_{w_j})$

This is the extra structure on the bundle.

Conversly, given an abstract collection of maps $\phi_{\beta\alpha}(x) : U_\alpha \cap U_\beta \to G$ satisfying $\phi_{\gamma\beta}(x) \cdot \phi_{\beta\alpha}(x) = \phi_{\gamma\alpha}(x)$ ↑ cocycle condition

for all α, β, γ and $\forall x \in U_\alpha \cap U_\beta \cap U_\gamma$

set $\tilde{E} = \coprod_{\alpha \in A} U_\alpha \times F \Big/ U_\alpha \times F \ni (x, v) \sim (x, \phi_{\beta\alpha}(x) \cdot v) \in U_\alpha \times F$

↑ glueing as trans. functions perscribe

with the obvious projection, get a bundle with structure G.

We get an isomorphic bundle if we start with one and forget the structure and then do $\tilde{E}$.

This construction makes it easy to do associated bundles. (in some way...)

Since G acts on itself by left translations ($G \times G \to G$), we can use the transition functions to construct a bundle with general fibre G (with acting on itself with left transl.) Such a bundle is called PRINCIPAL BUNDLE

Proposition. If $p : E \to B$ is a principle G-bundle, then G acts on E freely on the right with orbit space homeomorphic to B.

The principle bundle associated to a vector space is called the FRAME BUNDLE.

$$\begin{tikzcd} \mathbb{R}^n \arrow[d] & & GL(n,\mathbb{R}) \arrow[d] \\ E \arrow[d, "p"] \arrow[rr, squiggly, "\text{we take the frames}"] & & \tilde{E} \arrow[d] \\ B & & B \end{tikzcd}$$

Digression PULL-BACKS

Suppose given a diagram (in Top)

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

Define $P = \{(x,y) \in X \times Y \mid f(x) = g(y)\}$ as subspace of ~~[illegible]~~ the cartesian product $X \times Y$.
P is called the fibered product.

Suppose

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists!\varphi"] \arrow[drr, bend left, "\eta"] \arrow[ddr, bend right, "\xi"'] & & \\
& P \arrow[r] \arrow[d] & Y \arrow[d] \\
& X \arrow[r] & B
\end{tikzcd}$$

$\varphi(z) = (\xi(z), \eta(z))$

since the diagram is commutative, this lands in P. It is unique because we take the projections from P.

We say that
$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \\ X & \end{tikzcd}$$
is the LIMIT of the diagram
$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$

In this case, we call it a pull-back.

More generally given a diagram Γ in $\mathcal{C}$, a limit is a morphism $P \to \Gamma$, where $P \in \operatorname{Ob}\mathcal{C}$ such that given any other morphism $Z \to \Gamma$ ($Z \in \operatorname{Ob}\mathcal{C}$), there exists a unique morphism in $\mathcal{C}$ $Z \to P$ making everything commutative

morphisms to all objects

PULLBACKS OF BUNDLES

Fix a bundle $p: E \to B$. For a map $f: X \to B$ the associated map $\bar{p}: f^*E \to X$ is called the pullback of p along f.

$$\begin{tikzcd}
f^*E \arrow[r, "\bar{f}"] \arrow[d, "\bar{p}"] & E \arrow[d, "p"] \\
X \arrow[r, "f"] & B
\end{tikzcd}$$

Proposition. $\bar{p}$ is also a bundle.

Proof. Given fundamental $U_\alpha \subset B$, the obvious candidate in X is $f^{-1}(U_\alpha)$.

Trivialization $\{(y,e) \mid U_\alpha \ni f(y) = p(e)\} \longrightarrow f^{-1}(U_\alpha) \times F$

$(y,e) \longmapsto (y, pr_2 \Phi_\alpha(e))$

inverse: $(y, \Phi_\alpha^{-1}(f(y), v)) \longleftarrow (y, v)$

Exercise: Transition functions for the pullback are given by
$y \longmapsto \phi_{\beta\alpha}(f(y))$

(we get the contravariant action and that is why we have the asterisk upstairs)

Morphism of locally trivial fibered bundle with fibre F and structure group G.

Want commutative diagrams, where fibers are taken HOMEOMORPHICALLY to fibers preserving the structure:

$$\begin{tikzcd}
E \arrow[rr] \arrow[dr, red] \arrow[dd, "p"'] & & E' \arrow[dd, "p'"] \\
& f^*(E) \arrow[ur, red] \arrow[dl, red] & \\
B \arrow[rr, "f"] & & B'
\end{tikzcd}$$

this is the structure-preserving homeomorphism

Locally

$$\begin{tikzcd}
U \times F \arrow[r, "{(x,v) \mapsto (f(x), \psi_{UU'}(x)\cdot v)}"] \arrow[d] & U' \times F \arrow[d] \\
U \arrow[r, "f|_U"] & U'
\end{tikzcd}$$

, where $\psi_{UU'}: U \to G$

(we assume both U and U' are fundamental, if we have the maximal atlas we have this, otherwise we take some inclusion...)

Note. We can factor the diagram through the pullback and then we specify morphisms over the same base space, there we don't worry about $U \xrightarrow{f} U'$.

Classification theorem.

To every top. group G we can associate a "classifying space" BG together with a "universal" principal G-bundle $EG \to BG$ s.t. for any X, the set $[X, BG]$ is in bijective correspondence to iso. classes of principal G-bundles over X

homotopy classes of maps $X \to BG$

To $f: X \to BG$ associate the pullback $f^*(EG)$

$$\begin{tikzcd}
f^*(EG) \arrow[r] \arrow[d] & EG \arrow[d] \\
X \arrow[r, "f"] & BG
\end{tikzcd}$$

Given $EG \to BG$ (take any), then to $f: X \to BG$ associate the pullback $f^*(EG)$.

Example. $B\mathbb{Z} = S^1$ with e.g. $E\mathbb{Z} = \mathbb{R}$; $\exp: \mathbb{R} \to S^1$.

Liftings

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ {[0,1]} \arrow[r, "\gamma"'] \arrow[ur, dashed, "\Gamma"] & B \end{tikzcd}$$

Motivation in physics: motion and for $b \in B$, the associated F is the extra information (a matrix maybe)

$p(\Gamma(t)) = \gamma(t)$

Theorem. For a locally trivial fibre bundle p, a lifting above always exists.

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ W \arrow[r, "\gamma"'] \arrow[ur, dashed, "\Gamma"] & B \end{tikzcd}$$

given this commutative diagram, a lifting Γ makes everything commute

Proof. Using the Lebesgue covering lemma for some partition $0 = t_0 < t_1 < \dots < t_n = 1$ we have $\gamma([t_{i-1}, t_i]) \subset U_i$ for some fundamental U_i

For proof by induction suppose $\Gamma|_{[0, t_{i-1}]}$ has been constructed.

$$\begin{tikzcd} \{t_i\} \arrow[r, "\Gamma"'] \arrow[d] & E|_{U_i} \arrow[r, "\cong"'] \arrow[d] & U_i \times F \arrow[dl] \arrow[l, bend right, "\phi_i^{-1}(\gamma(t), \mathrm{const}_v)"'] \\ {[t_{i-1}, t_i]} \arrow[r, "\gamma"] & U_i & \end{tikzcd}$$

$U_i \times F \leftarrow (\gamma(t), \mathrm{const}_v)$

where $\phi_i(\Gamma(t_{i-1})) = (\gamma(t_{i-1}), v)$

By induction we have defined Γ on everything. $\square$

Lemma. If $i: Z \to W$ admits a retraction $r: W \to Z$ $(ri = id_Z)$, then for fundamental neighbourhood U, liftings in

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & E|_U \arrow[d] \\ W \arrow[r, "\gamma"'] & U \end{tikzcd}$$

always exist.

$$\begin{tikzcd} Z \arrow[r, "\zeta"] \arrow[d] & E|_U \arrow[r, "\cong", "\phi_U"'] \arrow[d] & U \times F \arrow[dl] \\ W \arrow[r, "\gamma"'] \arrow[ur, dashed, "\Gamma"] \arrow[urr, dashed, "\tilde{\Gamma}"] & U & \end{tikzcd}$$

$\tilde{\Gamma}(w) = (\gamma(w), pr_F\, \phi_U(\zeta(r(w))))$

$\Gamma(w) = \phi_U^{-1} \circ \tilde{\Gamma}$

$\Gamma(i(z)) = \phi_U^{-1}(\gamma(i z), pr_F\, \phi_U(\underbrace{\zeta r(i(z))}_{\zeta(z)}))$

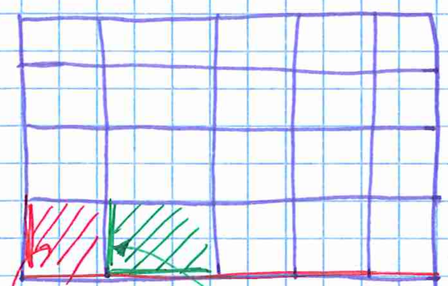
Corollary. For a fibre bundle $p: E \to B$ liftings in

$$\begin{tikzcd} {[0,1] \times \{0\}} \arrow[r] \arrow[d, hook] & E \arrow[d] \\ {[0,1] \times [0,1]} \arrow[r] & B \end{tikzcd}$$

always exist

lifting homotopies

Sketch of a proof: Using Lebesgue covering lemma we partition a square



partial lifting

lifting needs to coincide here on L, which is a retraction

Corollary. The same with

$$\begin{tikzcd} {[0,1]^n \times \{0\}} \arrow[r] \arrow[d] & E \arrow[d] \\ {[0,1]^n \times [0,1]} \arrow[r] & B \end{tikzcd}$$

or

$$\begin{tikzcd} Z \arrow[r, "\approx"] \arrow[d, "i"'] & {[0,1]^n \times \{0\}} \arrow[r] \arrow[d] & E \arrow[d] \\ W \arrow[r, "\approx"] & {[0,1]^n \times [0,1]} \arrow[r] & B \end{tikzcd}$$

B^n ... ball

Note, that $(B^n \times [0,1], B^n \times \{0\} \cup S^{n-1} \times [0,1]) \approx (B^n \times [0,1], B^n \times \{0\})$

exercise

Corollary can take $Z = K \times [0,1] \cup L \times [0,1]$, $W = L \times [0,1]$ for $K \le L$

lifting property (general)

cw complexes

5.3.2019

Definition. A map (continuous of course) $p: E \to B$ has <u>the homotopy lifting property</u> with respect to a space Z if every commutative square (in blue)

$$\begin{tikzcd} Z \arrow[r, "H_0"] \arrow[d, "{z \mapsto (z,0)}"'] & E \arrow[d, "p"] \\ Z \times [0,1] \arrow[r, "h"] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

(in topological sense, maps commute pointwise)

admits a filler $H: Z \times [0,1] \to E$

Def. A map $p: E \to B$ is called <u>Hurewicz fibration</u> if it has ~~the~~ homotopy lifting property with respect to ALL topological spaces Z.

(Note: we only call it fibration today)
We do this abstractly now.

Def. A map $p: E \to B$ has the <u>right lifting property</u> with respect to the map $i: Z \to W$ if every blue commutative square has a (green) filler.

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ W \arrow[r] \arrow[ur, dashed, "\text{lifting}"] & B \end{tikzcd}$$

The filler is not unique

Recall from last session, that a locally trivial fibre bundle has the homotopy lifting property with respect to disks B^n which forces the right lifting property with respect to inclusions $K \times [0,1] \cup L \times \{0\} \hookrightarrow L \times [0,1]$ for CW complexes $K \subseteq L$.

Theorem. If $p: E \to B$ is locally trivial fibre bundle over paracompact T_2 base B (e.g. for metrizable B), then it is a Hurewicz fibration.

No proof.

Example. Consider the space of continuous paths $[0,1] \to B$, which we'll denote $B^{[0,1]}$ $(= \mathcal{C}([0,1], B))$ with compact-open topology. (top space)
Then "evaluation at 0", assigning $\gamma(0)$ to $\gamma \in B^{[0,1]}$, is a fibration:

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & B^I \arrow[d] & \gamma \arrow[d, mapsto, "\text{a restriction}"] \\ Z \times [0,1] \arrow[r, "h"] \arrow[ur, dashed] & B = B^{\{0\}} & \gamma|_{\{0\}} \end{tikzcd}$$

$Z \times \{0\}$

(today $I = [0,1]$)

$(*)$

The diagram gives rise to maps $Z \to B^{I \times \{0\}}$ (from top level),
$Z \to B^{\{0\} \times I}$
$z \mapsto (\{0,t\} \mapsto h(z,t))$
$h(z,t)(0)$

the commutativity of diagram $(*)$ means, that the above maps actually define a map $Z \to B^{I \times \{0\} \cup \{0\} \times I}$

(the maps coincide on the intersection)

A filler in $(*)$ corresponds to a lifting of

$$\begin{tikzcd} & B^{I \times I} \arrow[d] \\ Z \arrow[r] \arrow[ur, dashed] & B^{I \times \{0\} \cup \{0\} \times I} \arrow[u, bend right, "s"'] \end{tikzcd}$$

(it is continuous in Z)

there is a section, because we have a retract of the square

the refraction

If $r: I \times I \to I \times \{0\} \cup \{0\} \times I$ is a retraction, we get a section map

$s: B^{I \times \{0\} \cup \{0\} \times I} \longrightarrow B^{I \times I}$ by (pre) composing with r.

<u>Note.</u> We actually do two adjunctions $Z \to B^I$, $Z \to B^{\{0\}}$ and then $Z \to B^{I \{0\}}$, $Z \to B^{\{0\} I}$

<u>Digression.</u> Let X, T, Y be spaces with T locally compact Hausdorff (e.g. T compact). Then the obvious map

$$\Phi: Y^{X \times T} \longrightarrow (Y^T)^X$$
$$\mathcal{C}(X \times T, Y) \longrightarrow \mathcal{C}(X, \mathcal{C}(T, Y))$$

is a homeomorphism

$$\Phi(h)(x) = (t \mapsto h(x,t))$$

to check: $\Phi(h)(x)$ is continuous, Φ is continuous in the compact open topology, the continuous inverse comes from T being locally compact.

Given T, I have obvious functors $L_T(X) = X \times T$ and $R_T(Y) = \mathcal{C}(T, Y)$

these are adjoint functors in the topological category, they form an adjoint pair.

<u>Example.</u> Let $K \subseteq L$ be CW complexes (we can just view polyhedra) (or finite simplicial complexes $\subset \mathbb{R}^N$).
Then for every B, the restriction map $B^L \to B^K$ is a fibration.

(We need to prove, that $K \times I \cup L \times \{0\}$ is a retract of $L \times I$)

this is a cannonical example.

Recall that $\pi_1(B, b_0) = [(S^1, *), (B, b_0)]$
↳ homotopical classes, homotopy respects base point

Generalising, $\pi_n(B, b_0) = [(S^n, *), (B, b_0)]$
(we omit the group structure)
(For $n \geq 2$, π_n is Abelian)

Theorem. (Whitehead) If $f: X \to Y$ induces isomorphisms on all π_n (all possible base points), then f is a homotopy equivalence, where X, Y are CW complexes. — or a warsaw circle (all homotopy groups are trivial) (and we can conclude) $\not\simeq \{*\}$
↳ Hawaiian earring does not have the type of a CW complex
(the other way is "obvious", if two spaces are homotopically equivalent, then they have same π_n)
This theorem is actually used in practice

Theorem. Let $p: E \to B$ be a fibration with $e_0 \in E$ over $b_0 \in B$ ($p(e_0) = b_0$). Let $F = p^{-1}(b_0)$. Then there is a long exact sequence of groups)

$$\to \pi_3(B, b_0) \to \pi_2(F, e_0) \to \pi_2(E, e_0) \to \pi_2(B, b_0) \to \pi_1(F, e_0) \to \pi_1(E, e_0) \to \pi_1(B, b_0)$$

Suppose we know that there is a fibration $\overset{\mathbb{C}^2}{S^3} \to S^2$ with fibers $\simeq S^1$

$(z, w) \mapsto [z : w] \in \mathbb{CP}^1$, $\frac{z}{w} \in \hat{\mathbb{C}}$
$|z|^2 + |w|^2 = 1$
one-point compactification

It is "easy" to show $(n \geq 1)$ $\pi_k(S^n) = \begin{cases} \{0\}, & k < n \\ \mathbb{Z}, & k = n \end{cases}$

$\pi_2(S^2) = \mathbb{Z}$

$\underbrace{S^1 \to S^3 \to S^2}_{\text{fibered bundle with fibre } S^1}$

If we omit base points we have an exact sequence

$$\underset{0}{\pi_3(S^1)} \to \underset{\mathbb{Z}}{\pi_3(S^3)} \xrightarrow{\cong} \underset{\mathbb{Z}}{\pi_3(S^2)} \to \underset{0}{\pi_2(S^1)} \to \dots$$

Also $\pi_k(S^1) = 0$ for $k \geq 1$

↳ this was very surprising

Properties.

- The pullback of a fibration is a fibration

Suppose the map p has the right lifting property to the fixed i

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "i"] & E \arrow[d, "p"] \\ W \arrow[r] & B \end{tikzcd}$$

right lifting property

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "i"] & P \arrow[r, "\bar{f}"] \arrow[d, "\bar{p}"] & E \arrow[d, "p"] \\ W \arrow[r] \arrow[ur, dashed] \arrow[urr] & X \arrow[r, "f"] & B \end{tikzcd}$$

we consider the outer rectangle ~~diag~~ and get the filler → + apply the pullback property

$$\begin{tikzcd} Z \arrow[r] & W \arrow[r, dashed] \arrow[rr, bend left] \arrow[dr, bend right] & P \arrow[r] \arrow[d] & E \arrow[d] \\ & & X \arrow[r] & B \end{tikzcd}$$

This diagram commutes, because we have a filler

then we make sure
$$\begin{tikzcd} Z \arrow[r] \arrow[d, "i"] & p \\ W \arrow[ur, dashed] & \end{tikzcd}$$
this commutes, we use the uniqueness of the pullback.

Homework: play with the diagram.

- The retract of a fibration is a fibration

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "i"] & E' \arrow[r, "J"] \arrow[d, "p'"] & E \arrow[r, "R"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ W \arrow[r] \arrow[urr, "f"] \arrow[urrr, "R \circ f"] & B' \arrow[r, "j"] & B \arrow[r, "r"] & B' \end{tikzcd}$$

Suppose p' is a retract of fibration $p: E \to B$

$\left.\begin{array}{l} R \circ J = id_{E'} \\ r \circ j = id_{B'} \end{array}\right\}$ retract condition

we just compose the filler f with R and get the desired filler

- "Any map is a fibration up to homotopy."

Theorem. Let $f: X \to B$ be a map
Then there exists a decomposition of f into $\underset{\text{homotopy equivalence}}{X \xrightarrow{s}} E \underset{\text{fibration}}{\xrightarrow{p}} B$,
(inclusion as a strong deformation retract)
which is cannonical
↳ [$s: X \to E$, $p: E \to B$ as objects in the category of maps are functorial in $f \in Map$]
↳ important!

Proof. pullback

$$\begin{tikzcd} X \sqcap B^I \arrow[r, "\bar{f}"] \arrow[d, "\bar{\varepsilon}_0"] & B^I \arrow[d, "\varepsilon_0"] \\ X \arrow[r, "f"] \arrow[u, bend left, "s"] & B \end{tikzcd}$$

ε_0: evaluation at 0

our E is the co mapping cylinder
$X \sqcap B^I = \{(x, \gamma) \in X \times B^I \mid f(x) = \gamma(0)\}$
Define $s: X \to X \sqcap B^I$ by (needs to be a section)
$s(x) = (x, const_{f(x)})$
and
$p: X \sqcap B^I \to B$ by
$p(x, \gamma) = \gamma(1)$

Obviously: s is a section.

We claim, that $s \circ \bar{\varepsilon}_0$ is $\simeq id$ (rel $s(X)$)
$s \circ \bar{\varepsilon}_0 (x, \gamma) = (x, const_{f(x)})$

let $X \sqcap B^I \times I \to X \times B^I$ be defined as
$((x, \gamma), t) \mapsto (x, \gamma_t)$
where $\gamma_t(u) = \gamma((1-t) \cdot u)$
(we think of it $\gamma|_{[0, 1-t]}$ reparametrised)

We can see, that on $s(X)$ this homotopy is stationary.
Exercise: p is a fibration. ☺ (easier after a bit more of this)

using that this is a fibration
Hint: can cheat by making use of fibration $B^I \to B \times B$ (eval of both endpoints)

Functoriality: a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ X' \arrow[r, "f'"] & B' \end{tikzcd}$$

can obviously be completed into a commutative prism

$$\begin{tikzcd} & E \arrow[dr] \arrow[dd] & \\ X \arrow[ur] \arrow[rr, "f"] \arrow[dd] & & B \arrow[dd] \\ & E' \arrow[dr] & \\ X' \arrow[ur] \arrow[rr, "f'"] & & B' \end{tikzcd}$$

EQUIVALENT FORMULATION OF FIBRATION

$$(*')\quad \begin{tikzcd} Z \arrow[r, "H_0"] \arrow[d] & E \arrow[d, "p"] \\ Z \times I \arrow[r, "h"] & B \end{tikzcd}$$

gives rise to a map $Z \longrightarrow B^I$, which together with $Z \longrightarrow E$ defines a map $Z \longrightarrow E \sqcap B^I$

$$E \sqcap B^I = \{(x,\gamma) \in E \times B^I \mid \gamma(0) = p(x)\}$$

$$\begin{tikzcd} Z \arrow[rr, "\hat{h}"] \arrow[dr] \arrow[dd, "H_0"'] & & B^I \arrow[dd] \\ & E \sqcap B^I \arrow[ur] \arrow[dl] & \\ E \arrow[rr, "p"] & & B \end{tikzcd}$$

Theorem: p is a fibration $\Longleftrightarrow$ the ~~map~~ cannonical map $E^I \to E \sqcap B^I$ has a section

$$\begin{tikzcd} E^I \arrow[r, "(\varepsilon_0, p_*)"] \arrow[rr, bend left, "p_*"] \arrow[dr, "\varepsilon_0"'] & E \sqcap B^I \arrow[r] \arrow[d] & B^I \arrow[d] \\ & E \arrow[r] & B \end{tikzcd}$$

p_* ... left pre-composing with p

Proof: ($\Leftarrow$) obvious, because lifting in $(*')$ is equivalent to that in

$$\begin{tikzcd} & E^I \arrow[d, "(\varepsilon_0, p_*)"] \\ Z \arrow[ur, dashed] \arrow[r] & E \sqcap B^I \arrow[u, dotted, bend right, "F"'] \end{tikzcd}$$

(we use the adjunction to E^I $\quad Z \times I \to E$) and the two components of the product give us the two triangles in the $(*')$ diagram

($\Rightarrow$) Use the lifting property: by adjunction we want to get a map $E \sqcap B^I \times I \longrightarrow E$; this will be a lifting of some map in B.

12.3.2019

Recall, that $p: E \to B$ is a fibration iff the cannonical map $E^I \longrightarrow E \sqcap B^I$, $\omega \longmapsto (\omega(0), p\circ\omega)$ has a section $\lambda: E \sqcap B^I \longrightarrow E^I$.
[To the pair (x,γ), where $x \in E$ and $p(x) = \gamma(0)$, λ associates a lifting beginning in x.]

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & x \arrow[d, dotted] \\ {[0,1]} \arrow[r, hook] & \gamma(0) \arrow[u, bend right, "\gamma"'] \end{tikzcd}$$

we call λ a LIFTING FUNCTION

we have fibers: preimages of points, we want to relate them to fibre bundles: they need to be nice spaces

FIBERS OF A FIBRATION

Fix a fibration $p: E \to B$ with lifting function λ.

Theorem: If $\gamma: [0,1] \longrightarrow B$ is a path, then fibers $F_0 = p^{-1}(\gamma(0))$ and $F_1 = p^{-1}(\gamma(1))$ are homotopy equivalent.

Proof: A map $\varphi: F_0 \longrightarrow F_1$ is given by
$$\varphi(x) = \underbrace{\lambda(x,\gamma)}_{\text{a lifting of } \gamma}(1)$$
← so it belongs to F_1

the inverse: $\psi(x) = \lambda(x, \bar{\gamma})(1)$
inverse time path

Homotopy?
Recall $\gamma_t(s) = \gamma((1-t)\cdot s)$ (shrinking γ)

$(x,t) \mapsto \lambda(\lambda(x,\gamma_t)(1), \bar{\gamma}_t)(1)$ ... one homotopy
is a homotopy $\psi \circ \varphi \simeq id_{F_0}$
(we use λ twice, to be in F_0 all the time, when we apply it to 1) $\square$

Exercise: do this without λ.

Obvious question:
For $\omega \in E^I$, can compare ω with $\lambda(\omega(0), p\circ\omega)$.
Actually,

Theorem. $E^I \longrightarrow E^I$
$\omega \longmapsto \lambda(\omega(0), p\circ\omega)$
is homotopic to id_{E^I} over $E \sqcap B^I$.

$$\begin{tikzcd} E^I \arrow[rr] \arrow[dr, "(\varepsilon_0, p_*)"'] & & E^I \arrow[dl, "(\varepsilon_0, p_*)"] \\ & E \sqcap B^I & \end{tikzcd}$$

Sketch of the proof:

At time t, associate to ω the concatenation $\omega|_{[0,t]}$ with $\lambda(\omega(t), \text{"}p\circ\omega|_{[t,1]}\text{"})$

prescribed beginning — we lift the remaining part

(we need to be careful to remain over ω) $\square$

Corollary. The maps $s: E \to E\Pi B^I$ (from the cannonical decomposition of $p: E\to B$; $s(x) = (x, \text{const}_{p(x)})$) and $\pi: E\Pi B^I \to E$, $\pi(x,\gamma) = \lambda(x,\gamma)(1)$

are mutually inverse HOMOTOPY EQUIVALENCES OVER B

$$\begin{tikzcd} & X\Pi B^I \arrow[r] \arrow[d] & B^I \arrow[d] \\ s: & X \arrow[r, "f"] & B \end{tikzcd}$$

each homotopy sits comutatively over B

$$\begin{tikzcd} E \arrow[rr, "s", "\simeq"'] \arrow[dr, "p"'] & & E\Pi B^I \arrow[ll, bend left, "\pi"] \arrow[dl, "\bar{\varepsilon}_1"] \\ & B & \end{tikzcd}$$

$\bar{\varepsilon}_1$ {induced evaluation at 1}

$$\begin{tikzcd} E \arrow[rr, "H_t"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} E\Pi B^I \arrow[rr, "K_t"] \arrow[dr, "\bar{\varepsilon}_1"'] & & E\Pi B^I \arrow[dl, "\bar{\varepsilon}_1"] \\ & B & \end{tikzcd}$$

(homotopy is fibered, each fibre lyes over B)

If we take a homotopy equivalence and do it in this decomposition we get a fibered homotopy

Consequently, the fibers of p and $\bar{\varepsilon}_1$ are homotopy equivalent.

Definition. For $f: X\to B$, the fibers of the cannonical fibration $\bar{\varepsilon}_1: X\Pi B^I \to B$ are called HOMOTOPY FIBERS of f.

(they are defined up to homotopy equivalence)

Explicitly, $\bar{\varepsilon}_1^{-1}(b_0) = \{(x,\gamma) \mid f(x) = \gamma(0),\ \gamma(1) = b_0\}$

(the geometric homotopy fiber)

PATH-LOOP FIBRATION

Let $b_0 \in B$. What do we get if we "turn" the inclusion $\{b_0\} \hookrightarrow B$ into a fibration?

$$\{b_0\}\Pi B^I \equiv \overbrace{\{\gamma \in B^I \mid \gamma(0) = b_0\}}^{PB = P(B,b_0)} \longrightarrow B$$
$$\gamma \longmapsto \gamma(1) = \varepsilon_1(\gamma)$$

the point b_0 is always first component,

PB is called the "path space" of based paths.

HW: check this is a fibration by def.

The fibration $PB \to B$ is called the PATH-LOOP fibration (after Serre)

Fibers?

Over b_1: $\{\gamma\in B^I \mid \gamma(0) = b_0,\ \gamma(1) = b_1\}$

over b_0: $\{\gamma \in B^I \mid \gamma(0) = b_0 = \gamma(1)\}$ – fiber over the distinguished base point is the loop space

Notation: $\Omega B \hookrightarrow PB \longrightarrow B$

Lemma. PB is contractible. The contraction $(\gamma, t) \mapsto \gamma_t \in PB$ $(\gamma_t(s) = \gamma((1-t)s))$

HW: define the constant path this maps to

Consider the pull-back

$$\begin{tikzcd} L_f \arrow[r] \arrow[d, "fib"] & PB \arrow[d] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

because pullback over fib is fib

$L_f = \{(x,\gamma) \mid \gamma(0) = b_0,\ f(x) = \gamma(1)\}$

This is cannonically (just inverse the time) homeomorphic to the homotopy fibre of $f: X\to B$ over b_0.

Discussion: Puppe sequence

Suppose we start with a fibration $E \xrightarrow{p} B$.

We also call L_f a homotopy fibre

fibre over cannonical point in B

$$\begin{tikzcd} F \arrow[d, "\cong"] \arrow[dr] & & \\ L_p \arrow[r, "2", "fib"'] & E \arrow[r, "p"] & B \end{tikzcd}$$

$$\begin{tikzcd} L_q \arrow[r, "12"] \arrow[d, "\cong"] & L_p \arrow[r, "2"] & E \\ \Omega B \arrow[r] & L_p & \end{tikzcd}$$

we can view up to homotopy:

$\Omega E \to \Omega B \to F \to E \to B$

(we have to have compatible base points $p(e_0) = b_0$)

and now we continue this to ∞: Puppe sequence

$$\cdots \to \Omega^2 E \to \Omega^2 B \to \Omega F \to \boxed{\Omega E \to \Omega B \to F \to E \to B}$$

loop space of a loop space, the base point is the constant path at b_0.

Lemma. If $F \hookrightarrow E \to B$ is a fibration and X is any space, the associate sequence

$$[X,F] \to [X,E] \to [X,B]$$ is exact at $[X,E]$.

(if $X = S^0$, then we get a long homotopy exact sequence of a fibration)

We need to switch to homotopy classes, that preserve the base point.

COFIBRATIONS

Names: Hurewicz "fibrations" 1955 (Hurewicz)
"cofibration" 1958 (Eckmann)
but the notions have been here before the names

Def. The pair (X, A) of top. spaces ($\in \mathrm{Ob}\,\mathrm{Top}^2$, $A \subseteq X$) has the homotopy extension property w.r.t. Y if for every map $f: X \to Y$ and every (partial) homotopy $h: A \times I \to Y$ with $h(a,0) = f(a)$ for all $a \in A$ there exists an EXTENSION $H: X \times I \to Y$ extending both.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Y^I \arrow[d, "\varepsilon_0"] \\ X \arrow[r, "f"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

In other words, i has the LEFT LIFTING PROPERTY w.r.t. $\varepsilon_0: Y^I \to Y$.

Obviously we can generalise to an arbitrary map $i: A \to X$.

Def. The map $i: A \to X$ in Top is a cofibration if it has the left lifting property w.r.t. $Y^I \to Y$ for all Y.

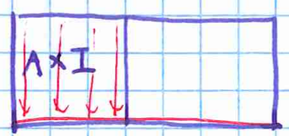
This concept was around in 1930s.

We dualize (by Eckmann-~~Hilton~~ dual), we get the homotopy lifting property

$$\begin{tikzcd} B & Z \times I \arrow[l] \arrow[dl, dashed] \\ E \arrow[u] & Z \arrow[l] \arrow[u, "i_0"'] \end{tikzcd}$$

Theorem (Arne Strøm) If $i: A \to X$ is a cofibration, then i is a topological embedding.

Theorem. (A. Strøm) For $A \subseteq X$ the inclusion $i: A \hookrightarrow X$ is a cofibration iff $A \times [0,1] \cup X \times \{0\}$ is a retract of $X \times [0,1]$



Proof. (in the special case A closed) (so $f \cup h$ are continuous)

($\Longrightarrow$) let $i: A \to X$ be a cofibration.

$$\begin{tikzcd} A \times [0,1] \cup X \times \{0\} \arrow[r, "id"] \arrow[d] & \underbrace{A \times [0,1] \cup X \times \{0\}}_{Y} \\ X \times [0,1] \arrow[ur, dashed] & \end{tikzcd}$$

since it is prescribed to be id on the subspace, this is a retraction.

($\Longleftarrow$) The f and h define a continuous map $A \times [0,1] \cup X \times \{0\} \xrightarrow{h \cup f} Y$, can compose with the retraction.

The general case it is difficult. □

Example. For any space Z the $Z \equiv Z \times \{0\} \hookrightarrow Z \times [0,1]$ is a cofibration

Proof:

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y^I \arrow[d, "\varepsilon_0"] \\ Z \times [0,1] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

this is a fibration, so it has the right lifting property

ε_0 fibration? ∃ retraction $r: [0,1] \times [0,1] \to \{0\} \times I \cup I \times \{0\}$.

Retraction $Z \times I \times I \to Z \times \{0\} \times I \cup Z \times I \times \{0\}$
(we keep Z fixed and on the remaining we apply the same retraction) □

Digression: PUSHOUT

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d] \arrow[ddr, bend left] & \\ X \arrow[r] \arrow[drr, bend right] & A \sqcup_f B \arrow[dr, dashed, "\exists!"] & \\ & & Y \end{tikzcd}$$

colimit

dual to the pullback:

$$\begin{tikzcd} Z \arrow[r, dashed] \arrow[rr, bend left] \arrow[dr, bend right] & X \sqcap E \arrow[r] \arrow[d] & E \arrow[d] \\ & X \arrow[r] & B \end{tikzcd}$$

limit

Given diagram $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \\ X & \end{tikzcd}$ we have an inductive colimit, such that if we have compatible maps there is a unique map out of the colimit.

Proposition. Cofibrations are closed under pushouts and retractions.

Proof:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"', "\text{cofib.}"] & B \arrow[r, "h"] \arrow[d, "\varphi"] & Y^I \arrow[d, "\varepsilon_0"] \\ X \arrow[r] \arrow[ur, dashed, "\text{a filler from } i"] & Q \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

the induced map of the pushout is also a cofib

now φ and h by pushout property define the unique map from Q to Y^I, that satisfy.

For retractions:

$$\begin{tikzcd} A' \arrow[r, "j"] \arrow[d, "i'"'] & A \arrow[r, "r"] \arrow[d, "i"] & A' \arrow[r] \arrow[d] & Y^I \arrow[d] \\ B' \arrow[r, "j"'] \arrow[ur, phantom, "\text{an extra step}"] & B \arrow[r, "R"'] \arrow[ur, phantom, "\text{lifting of } i"] & B' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

retraction of i

$rj = id_{A'}$
$Rj = id_{B'}$

A dual of a retraction is a retraction

19.3.2019

Recall, that $i: A \to X$ is a cofibration if it has the left lifting property w.r.t. $\varepsilon_0: Y^I \to Y$ for all Y

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Y^I \arrow[d, "\varepsilon_0"] \\ X \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

And recall that if i is a cofibration, then it's an embedding (of top spaces).

Theorem. (Strøm) If $A \subseteq X$ is a cofibration, so is $\bar{A} \subseteq X$.

(we will often consider just closed ones)

Recall that $A \subseteq X$ is a cofibration iff $A \times I \cup X \times \{0\}$ is a retract of $X \times I$

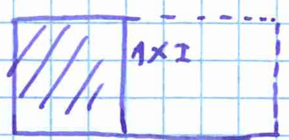


figure out why this is functorially (the union $X\times I \amalg Y$ is [illegible])

Theorem. Every map can be factored (functorially) into the composite of a cofibration followed by a homotopy equivalence.

Proof.

Duality with fibrations

$$\begin{tikzcd} X \sqcap B^I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B^I \arrow[d, "\varepsilon_0"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

pullback

(evaluation at 1 is the fib)

$\rightsquigarrow$

$$\begin{tikzcd} X\times I \sqcup_f Y & X \times I \arrow[l] \\ Y \arrow[u] & X \arrow[l] \arrow[u, "i_0"'] \end{tikzcd}$$

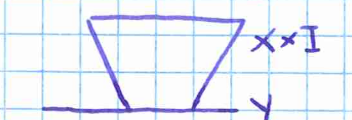
pushout

i_0 (inclusion at 0)

We call the adjunction space

$$M_f = \frac{X \times I \amalg Y}{(x,0) \sim f(x)}$$

the MAPPING CYLINDER



$\bar{\varepsilon}_1: X \sqcap B^I \longrightarrow B$ $\rightsquigarrow$ $\bar{i}_1: X \longrightarrow M_f$

(inclusion into the top level of the mapping cylinder, nothing happens)

$s: X \longrightarrow X \sqcap B^I$

$x \longmapsto (x, \mathrm{const}_{f(x)})$ $\rightsquigarrow$ $\pi: M_f \longrightarrow Y$

$[x,s] \longmapsto f(x)$

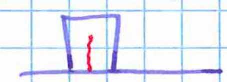
$[y] \longmapsto y$ (if f not surj.: we need this)

- π is a homotopy equivalence:

  Y is obviously a retract of M_f (we just include Y into $X \times I \amalg Y$, followed by the quotient map)

  Strong deformation retraction (SDR): $([x,s],t) \longmapsto [x, (1-t)s + t\cdot 0]$

  $([y],t) \longmapsto [y]$

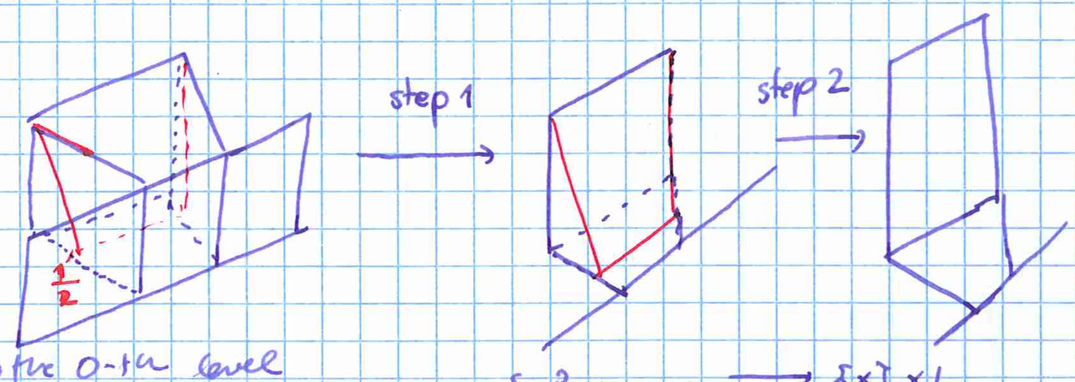


so we have a homotopy equiv.

- $X \equiv X \times \{1\} \hookrightarrow M_f$ is a (closed) fibration

  we have to produce a retraction $M_f \times I \longrightarrow X \times \{1\} \times I \cup M_f \times \{0\}$

$M_f \times I$

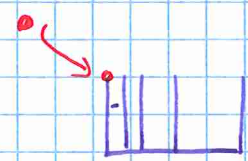


(we project to the 0-th level of $M_f \times \{0\}$ the first half)

$\{x\} \times$ triangle $\longrightarrow \{x\} \times$ ∟

Examples. For $K \subseteq L$ CW complexes, $K \subseteq L$ is a cofibration

Non Example. The inclusion



to a top. comb is not a cofibration

if we take $t > 0$ and a point close to "the whisker", because of continuity, a very short path to a very long path down.

For a pointed space (X, x_0) we say, that x_0 is non-degenerate if $\{x_0\} \hookrightarrow X$ is a closed fibration.

Lemma. Suppose $f: X \longrightarrow Y$ is a map, where $f(x_0)$ lies in the same path component as $y_0 \in Y$.

If x_0 is non-degenerate, then f is homotopic to $\tilde{f}$ sending x_0 to y_0

Proof.

$\{x_0\} \times I$ $\longrightarrow$ a path from $f(x_0)$ to y_0

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f \text{ on the rest}"] & Y \\ X \times I \arrow[u, "\text{retr.}"] \arrow[ur, "h"'] & \end{tikzcd}$$

homotopy level 1 of h is $\tilde{f}$.

$\square$

A digression on duality:

Why is $i_0 : X \longrightarrow X \times I$ dual to $\varepsilon_0 : Y^I \longrightarrow Y$?

Adjunction:

$(X\times I \to Y) \longrightarrow (X \to Y)$, $f \longmapsto (x \mapsto f(x,0))$

$$\begin{tikzcd} \mathcal{C}(X, Y^I) \arrow[r, phantom, "\cong"] \arrow[d, "(\varepsilon_0)_*"] & \mathcal{C}(X\times I, Y) \arrow[d, "(i_0)_*"] \\ \mathcal{C}(X,Y) \arrow[r, phantom, "\cong"] & \mathcal{C}(X,Y) \end{tikzcd}$$

(an adjunction of two functors)

this one has to be the inclusion at 0

a prolongation of evaluation at 0

What is the dual of $Y \longrightarrow Y \times Y$, $y \longmapsto (y_0, y)$?

$$\begin{tikzcd} & Y \arrow[r] & Y\times Y \\ & X \arrow[u, bend left] \arrow[ur, bend right] & \end{tikzcd}$$

$(X \to Y) \longrightarrow (X \sqcup X \to Y)$

f ; $\iota_1(x) \longmapsto y_0$? ; $\iota_2(x) \longmapsto f(x)$

↳ this depends on f, so it is not a representation!

$$\begin{tikzcd} \mathcal{C}(X, Y\times Y) \arrow[r, phantom, "\cong"] & \mathcal{C}(X \sqcup X, Y) \\ \mathcal{C}(X,Y) \arrow[u] \arrow[r, phantom, "\cong"] & \mathcal{C}(X,Y) \arrow[u] \end{tikzcd}$$

post-compose here

pre-compose here

↳ a map to a cartesian product is two maps, so a map from disj. union

$X \sqcup X \longrightarrow X$

this map is not induced by a fixed map $X \sqcup X \longrightarrow X$, so this does not have a dual.

$f : X \to Y \rightsquigarrow x \longmapsto (y_0, f(x))$

We would need a base point for this to work.
we are trying to refer to y_0, but we cannot without base point.

Not every diagram can be dualised, so we do not have the same properties for cofibrations as we have for fibrations.

Recall the concept of fibre (pre-image of a point)

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ \{b_0\} \arrow[r, hook] & B \end{tikzcd}$$

and the homotopy fibre

$$\begin{tikzcd} L_f \arrow[r] \arrow[d] & PB \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$

whose

$$\begin{tikzcd} PB \arrow[r] \arrow[d] & B^I \arrow[d, "(\varepsilon_0, \varepsilon_1)"] \\ B \arrow[r] & B\times B \end{tikzcd}$$

$x \longmapsto (b_0, x)$

we don't have the duals because of the base points

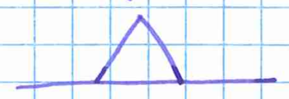
For cofibrations

$$\begin{tikzcd} X \sqcup_i \{*\} & X \arrow[l] \\ \{*\} \arrow[u] & A \arrow[l] \arrow[u, "i"] \end{tikzcd}$$

$X \sqcup_i \{*\} \cong X/A$

The "dual" of the path space PB is the cone $X \times I / X \times \{1\}$ (CX) and the $\boxed{\text{homotopy cofibre}}$ is the adjunction space

$C_f = CX \sqcup_{(x,0) \sim f(x)} Y$ (mapping cone)



(pointed hat)

There is a "dual" Puppe sequence

$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X \longrightarrow \cdots$

space ; codomain ; mapping cone on the map

we get a long sequence

QUILLEN'S MODEL CATEGORY

Let $\mathcal{C}$ be a category with three distinguished classes of morphisms (maps), called COFIBRATIONS ($\mathcal{CF}$), FIBRATIONS ($\mathcal{F}$), WEAK EQUIVALENCES ($\mathcal{WE}$)

(Quillen, 1967)

Then $\mathcal{C}$ is a (closed) model category if it satisfies

MC 1 $\mathcal{C}$ has finite limits and colimits

MC 2 If in a commutative triangle in $\mathcal{C}$ any two arrows are $\mathcal{WE}$, so is the third (2-out-of-3 property)

$$\begin{tikzcd} \bullet \arrow[rr] \arrow[dr] & & \bullet \\ & \bullet \arrow[ur] & \end{tikzcd}$$

MC 3 Classes $\mathcal{CF}$, $\mathcal{F}$, $\mathcal{WE}$ are closed under retracts

MC 4 In

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

assume $i \in \mathcal{CF}$ and $p \in \mathcal{F}$.

There exists a lifting if, in addition, $i \in \mathcal{WE}$ or $p \in \mathcal{WE}$.

Recall we had

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & Y^I \arrow[d, "\varepsilon_0"] \\ \bullet \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r] \arrow[d] & \bullet \arrow[d] \\ Z\times I \arrow[r] & \bullet \end{tikzcd}$$

, these are the generators for us (see theorem later)

MC5 Every morphism f admits factorisations $f = p\,i = q\,j$, where i, j are cofibrations, p, q are fibrations and i, q are in $\mathcal{WE}$.

Def. A TRIVIAL (CO)FIBRATION is a (co)fibration, that is in $\mathcal{WE}$

Theorem. (Strøm, 1972) Top with CLOSED cofibrations, fibrations and (honest) homotopy equivalences is a model category.

(we have not proved MC4, and MC5 the trivial part)
↳ difficult ↳ but it can be done.

Immediate corollaries. (of the definition)

"CLOSED"

- $\mathcal{CF} = \{ i \mid i$ have left lifting property w.r.t. trivial fibrations$\}$
  $\supset$ (closed under this construction)
  $\mathcal{CF} \cap \mathcal{WE} = \{ i \mid i$ have left lifting property w.r.t. all fibrations$\}$
- $\mathcal{F} = \{ p \mid p$ have the right lifting property w.r.t. trivial cofibrations$\}$
  $\mathcal{F} \cap \mathcal{WE} = \{ p \mid \ldots$ right lifting property all cofibrations

Proof $\mathcal{CF} \cap \mathcal{WE} = \{\ldots\}$

Let $i : A \to X$ have the left lifting property w.r.t. all fibrations.

Factor
$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, tail, "\sim"'] & & X \\ & A' \arrow[ur, two heads] & \end{tikzcd}$$

legend: $\rightarrowtail$ cofib, $\xrightarrow{\sim}$ we, $\twoheadrightarrow$ fib

Lift
$$\begin{tikzcd} A \arrow[r, tail, "\sim"] \arrow[d] & A' \arrow[d, two heads] \\ X \arrow[r, equal] \arrow[ur, dashed] & X \end{tikzcd}$$

by LLP, that is assumed.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "i"] & A \arrow[r, equal] \arrow[d, "\sim"] & A \arrow[d] \\ X \arrow[r] & A' \arrow[r] & X \end{tikzcd}$$

these are the two commutative △

i is a retract of a map $\in \mathcal{CF} \cap \mathcal{WE}$ and therefore also $i \in \mathcal{CF} \cap \mathcal{WE}$ by MC3. □

"ISOMORPHISMS, PUSHOUTS, PULLBACKS"

- $\mathcal{CF}$ are closed under pushouts } in the Top the proof was abstract,
  $\mathcal{F}$ —||— pullbacks } just used the lifting properties
- ISOMORPHISMS $\subset \mathcal{CF} \cap \mathcal{F} \cap \mathcal{WE}$
  (isomorphism has an inverse, that gives the other comm. square)

INITIAL AND TERMINAL OBJECTS
↳ colimit of the empty diagram, we denote it $\emptyset$.

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[dr] & Z \arrow[d, "\exists !"] \\ & Y \end{tikzcd}$$

empty diagram: no commutativity relations, no condition

Dual $Z \xleftarrow{\exists !} Y$ terminal object will be denoted by $*$ (like a generic point, singleton)

Initial and terminal objects exist because finite limits exist.

26.3.2019

HOMOTOPY AND THE HOMOTOPY CATEGORY

Fix a model category with initial object $\emptyset$ (unique up to isomorphism) and terminal object $*$.

Definition. X is called COFIBRANT if the unique $\emptyset \to X$ is a cofibration. Dually, Y is FIBRANT if the unique map $Y \to *$ is a fibration.

Note, that Top with Hurewicz model structure is such that all objects are both fibrant and cofibrant.

Exercise. Let U be a fixed object in $\mathcal{C}$ and let $U \downarrow \mathcal{C}$ be the category "under" U (objects maps to U) (the coslice category under U)

$$\begin{tikzcd} & U \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

By letting f be a cofibration (resp. fibration, w.e.) in $U \downarrow \mathcal{C}$ iff f is a cofibration (resp. fibration, w.e.) in $\mathcal{C}$, we get a model structure on $U \downarrow \mathcal{C}$.

Example. Taking $U = \{*\}$ (one-point space), $\{*\} \downarrow$ Top is just a category of pointed spaces.

$$\begin{tikzcd} & \{*\} \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"] & & Y \\ x_0 \arrow[rr, dashed, mapsto] & & y_0 \end{tikzcd}$$

Taking the "inherited" model structure, precisely those (X, x_0) are cofibrant for which $\{x_0\} \hookrightarrow X$ is a closed cofibration.

$\{*\} \downarrow$ Top is a pointed category.

Definition. Given X, a cylinder for X is a commutative diagram

$$\begin{tikzcd} X \sqcup X \arrow[d, "i"'] \arrow[dr, "\nabla = id + id"] & \\ \tilde{X} \arrow[r, "p"'] & X \end{tikzcd}$$

coproduct; fold map

where i is a cofibration and p is a weak equivalence

Digression. Notation.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & Y \arrow[d, "i_R", "i_Y"'] \arrow[ddr, bend left, "g"] & \\ X \arrow[r, "i_L", "i_X"'] \arrow[drr, bend right, "f"'] & X \sqcup Y \arrow[dr, dashed, "f+g"] & \\ & & Z \end{tikzcd}$$

left map; right map; (the coproduct)

the diagram commutes, since $\emptyset$ is the initial object and there is only one map $\emptyset \to Z$

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d, "i_R"] \arrow[ddr, bend left, "id"] & \\
X \arrow[r, "i_L"] \arrow[drr, bend right, "id"'] & X \sqcup X \arrow[dr, dashed, "\nabla = id + id"] & \\
& & X
\end{tikzcd}$$

for $\xi : X \to X'$ and $\eta : Y \to Y'$ we'll use $\xi \sqcup \eta = i_L \xi + i_R \eta$

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left, "i_R \eta"] & \\
X \arrow[r] \arrow[drr, bend right, "i_0 \xi"'] & X \sqcup Y \arrow[dr, dashed, "\xi \sqcup \eta"] & \\
& & X' \sqcup Y'
\end{tikzcd}$$

$$\begin{tikzcd}
X \times \{0,1\} = X \sqcup X \arrow[d, "i"'] \arrow[dr, "\nabla"] & \\
X \times [0,1] \arrow[r, "p"] & X
\end{tikzcd}$$

$i_0 + i_1 = i$

inclusion on 0-th level

i is a cofibration

Definition. Let $f, g : X \to Y$ be maps. A left homotopy $f \to g$ with respect to $\tilde{X}$ (the entire comm. triangle) is a commutative triangle

$$\begin{tikzcd}
X \sqcup X \arrow[d, "i_0 + i_1"'] \arrow[dr, "f+g"] & \\
\tilde{X} \arrow[r, "h"] & Y
\end{tikzcd}$$

it is the usual homotopy

$$\begin{tikzcd}
X \times \{0,1\} \arrow[d] \arrow[dr, "f+g"] & \\
X \times [0,1] \arrow[r, "h"] & Y
\end{tikzcd}$$

We call f and g left homotopic if a left homotopy exists w.r.t. SOME CYLINDER.

Lemma. Suppose X is cofibrant.
(1) If $\tilde{X}$ is a cylinder, then i_0 and i_1 are trivial cofibrations.
(2) Left homotopy is an equivalence relation on $\mathrm{Hom}(X,Y) = \mathcal{C}(X,Y)$.

Note: In top this seems trivial, since i_0 and i_1 are just inclusions, so the interval $[0,1]$ is contractible and i_0 and i_1 are weakly equiv.

Proof.
(1)
$$\begin{tikzcd}
\emptyset \arrow[r, tail] \arrow[d, tail] & X \arrow[d, "i_R"] \\
X \arrow[r, tail, "i_L"] & X \sqcup X
\end{tikzcd}$$

Since X is cofibrant i_L and i_R are pushouts of cofibrations and are themselves such.

Clearly, $i_0 = i \circ i_L$ (see diagram that defines $i_0 + i_1$) and $i_1 = i \circ i_R$ are composits of cofibrations.

The defining triangle for cylinder yields

$$\begin{tikzcd}
X \arrow[d, "i_0"'] \arrow[dr, "id"] & \\
\tilde{X} \arrow[r, "p"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, "i_1"'] \arrow[dr, "id"] & \\
\tilde{X} \arrow[r, "p"] & X
\end{tikzcd}$$

id is an iso, so w.e., p is w.e., by 2-out-of-3, i_0 and i_1 are weak equivalences.

(2) Reflexivity: suppose $f : X \to Y$

$$\begin{tikzcd}
X \sqcup X \arrow[d] \arrow[dr, "\nabla"] \arrow[drr, "f+f"] & & \\
\tilde{X} \arrow[r] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

symmetry: In topology, take $h(x, 1-t)$. (we take 0-th level to the 1st and vice versa. we now do it abstractly)
Abstractly, we have to switch copies in $X \sqcup X$

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d, "i_R"] \arrow[ddr, bend left, "i_L"] & \\
X \arrow[r, "i_L"] \arrow[drr, bend right, "i_R"'] & X \sqcup X \arrow[dr, "\tau"] & \\
& & X \sqcup X
\end{tikzcd}$$

cylinder:
$$\begin{tikzcd}
X \sqcup X \arrow[d, "i \circ \tau"'] \arrow[dr, "\nabla"] & \\
\tilde{X} \arrow[r, "p"] & X
\end{tikzcd}$$

pre-composing i with τ will remain a cylinder object

homotopy:
$$\begin{tikzcd}
X \sqcup X \arrow[d, "i \circ \tau"'] \arrow[dr, "(f+g)\tau = g+f"] & \\
\tilde{X} \arrow[r, "h"] & Y
\end{tikzcd}$$

transitivity: In topology $h : X \times [0,1] \to Y$ $\quad f_0 \rightsquigarrow f_1$
$\tilde{h} : X \times [0,1] \to Y$ $\quad f_1 \rightsquigarrow f_2$

$\boxed{\begin{matrix} X \times [\frac{1}{2}, 1] \\ X \times [0, \frac{1}{2}] \end{matrix}}$ ← glueing

Suppose we have cylinder objects

$$\begin{tikzcd}
X \sqcup X \arrow[d, "i"'] \arrow[dr, "\nabla"] & \\
\tilde{X} \arrow[r, "p"] & X
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \sqcup X \arrow[d, "i'"'] \arrow[dr, "\nabla"] & \\
\tilde{X}' \arrow[r, "p'"] & X
\end{tikzcd}$$

Define
$$\begin{tikzcd}
X \arrow[r, tail, "i_0'", "\sim"'] \arrow[d, "i_1"'] & \tilde{X}' \arrow[d, "j'"] \arrow[ddr, bend left, "p'"] & \\
\tilde{X} \arrow[r, tail, "j", "\sim"'] \arrow[drr, bend right, "p"'] & \bar{X} \arrow[dr, "\bar{p}"] & \\
& & X
\end{tikzcd}$$

trivial cofib by (1)

j is the pushout of the trivial cofibration i_0', hence a w.e.
By 2-out-of-3 $\bar{p}$ is a w.e.

We define $\bar{i} := j \circ i_0 + j' \circ i_1'$

clearly, $\bar{p}\,\bar{i} = \bar{p} j i_0 + \bar{p} j' i_1 = \underbrace{p i_0}_{id} + \underbrace{p' i_1'}_{id} = \nabla$

$$\begin{tikzcd}
X \sqcup X \arrow[d, "\bar{i}"'] \arrow[dr, "\nabla"] & \\
\bar{X} \arrow[r, "\bar{p}"'] & X
\end{tikzcd}$$

we get the cylinder this way.

Now we need the homotopy.

Consider

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d] \\
\emptyset \arrow[r] & X
\end{tikzcd}$$

we combine the two pushouts

$$\begin{tikzcd}
X \arrow[r, "i_L\, id"] \arrow[d, "i_0"'] & X \sqcup X \arrow[d, "i_0 \sqcup id"] \\
\tilde{X} \arrow[r, "i_L\, id"'] & \tilde{X} \sqcup X
\end{tikzcd}$$

(abstract pushout, add a coproduct, it'll still be a pushout: exercise)

$$\begin{tikzcd}
X \sqcup X \arrow[r, "i_0' + i_1' = i'"] \arrow[d, "i_1 \sqcup id"'] & \tilde{X}' \arrow[d, "j'"] \\
\tilde{X} \sqcup X \arrow[r, "j + j' i_1'"'] & \bar{X}
\end{tikzcd}$$

(exercise: abstract pushout, coproduct with id, add commutative triangle

$$\begin{tikzcd}
X \arrow[r, "i_1'"] \arrow[dr, "j' i_1'"'] & \tilde{X}' \arrow[d, "j'"] \\
& \bar{X}
\end{tikzcd}$$

, it is still a pushout)

since i_0 is cofib, so is $i_0 \sqcup id$

similarly from $i' \Rightarrow j + j' i_1'$

Note, that $\iota = (j + j' i_1') \circ (i_0 \sqcup id)$.

Now we glue the homotopies. □

Right homotopy

A **path object** for Y is a commutative triangle

$$\begin{tikzcd}
& \hat{Y} \arrow[d, "(\varepsilon_0, \varepsilon_1)"] \\
Y \arrow[ur, "c"] \arrow[r, "\Delta"'] & Y \times Y
\end{tikzcd}$$

, where $(\varepsilon_0, \varepsilon_1)$ is a fibration and c is a weak equivalence.

$$\begin{tikzcd}
& Y^I \arrow[d, "(\varepsilon_0, \varepsilon_1)"] \\
Y \arrow[ur, "c"] \arrow[r, "\Delta"'] & Y \times Y
\end{tikzcd}$$

c ... constant inclusion

Δ ... diagonal

$$\boxed{\varepsilon_0 \circ c = \varepsilon_1 \circ c = id}$$

Def. A right homotopy $f \longrightarrow g$ w.r.t. a given path object $\hat{Y}$ is a commutative diagram (where $f, g : X \longrightarrow Y$)

$$\begin{tikzcd}
& \hat{Y} \arrow[d, "(\varepsilon_0, \varepsilon_1)"] \\
X \arrow[ur, "k"] \arrow[r, "(f,g)"'] & Y \times Y
\end{tikzcd}
\quad \underset{\sim}{\text{in Top}} \quad
\begin{tikzcd}
& Y^I \arrow[d] \\
X \arrow[ur, "k"] \arrow[r] & Y \times Y
\end{tikzcd}$$

Def: f and g are right homotopic if $\exists$ right homotopy w.r.t. SOME $\hat{Y}$.

Lemma. Suppose Y is fibrant

(1) $\varepsilon_0, \varepsilon_1$ are trivial fibrations

(2) right homotopy is an equivalence relation on $Hom(X, Y)$, for any X.

We form the opposite category (inverting the arrows), takes cofibrations to fibrations and w.e to w.e.

We get the oposite model category

Proof. By duality using $\mathcal{C}^{op}$.

Proposition. Suppose X is cofibrant and $\hat{Y}$ is a given fixed path object. If $f, g : X \longrightarrow Y$ are left homotopic, they are right homotopic w.r.t. $\hat{Y}$.

Theorem. Let X be cofibrant and Y fibrant. The following are equivalent:

prop (1) f, g are left homotopic

triv. (2) f, g are right homotopic w.r.t. a fixed path object

dual of the prop (3) f, g are right homotopic

(4) f, g are left homotopic w.r.t. a fixed cylinder.

Proof. $(1) \xRightarrow{\text{prop}} (2) \xRightarrow{\text{trivial}} (3) \xRightarrow{\text{dual of prop}} (4) \xRightarrow{\text{triv.}} (1)$ □

Note. For proving, sometimes right homotopy is better, sometimes right is better.

Definition. Let $\Pi\mathcal{C}_{cf}$ be the category with objects, that are cofibrant-fibrant objects of model cat. $\mathcal{C}$ and morphisms homotopy classes of morphisms (all notions of homotopy are equiv. by Theorem)

This is (a version of) THE HOMOTOPY CATEGORY.

(we had for Hurewicz case all this)

Abstract localisations

(similar to rings localisation)

$\mathbb{Z}$, prime ideal (p)

$\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}$

$\mathbb{Z}[\frac{1}{p}] \longrightarrow \mathbb{Z}[\frac{1}{p}]$

Def. In a category $\mathcal{C}$ suppose given a class of morphisms S (closed under composition). The localisation of $\mathcal{C}$ w.r.t. S is a category "obtained" by forcing the elements of S to be isomorphisms. Formally, this is a category $S^{-1}\mathcal{C}$ with a functor $\mathcal{C} \xrightarrow{\Gamma} S^{-1}\mathcal{C}$ [with the following universal property: taking elements of S to isomorphisms

If $G : \mathcal{C} \longrightarrow \mathcal{D}$ is any functor taking S to isomorphisms, there exists a unique functor $F : S^{-1}\mathcal{C} \longrightarrow \mathcal{D}$ making

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "G"] \arrow[dr, "\Gamma"'] & \mathcal{D} \\
& S^{-1}\mathcal{C} \arrow[u, dashed, "\exists! F"']
\end{tikzcd}$$

Def. $Ho(\mathcal{C}) := (W\mathcal{C})^{-1}\mathcal{C}$.
[with respect to weak equivalences

We force we to be iso.

Note, that the universal functor $\mathcal{C} \to Ho(\mathcal{C})$ takes weak equivalences to ISOMORPHISMS.

Theorem. (Quillen) There is an "obvious" functor $\mathcal{C} \to \pi\mathcal{C}_{cf}$ exhibiting the equivalence $\pi\mathcal{C}_{cf} \cong Ho(\mathcal{C})$.
[equivalence of categories

Comments. You have to assign to each object $X \in \mathcal{C}$ an object, which is cofibrant-fibrant, which is weakly equivalent to X.

Factor $\emptyset \to X$ into $\emptyset \rightarrowtail QX \overset{\sim}{\twoheadrightarrow} X$ ↳ we consider the universal property
↳ model cat. structure
[QX is cofibrant and weakly equivalent to X.

and $QX \to *$ into $QX \overset{\sim}{\rightarrowtail} RQX \twoheadrightarrow *$,
so RQX is cofibrant fibrant

This correspondence has to be made functorially (we can always do it functorially, how we do it depends on $\mathcal{C}$)
(we have to define $QX = X$ if X is cofibrant and $RQX = QX$ if QX is fibrant)

Rather $Y \to *$ into $Y \overset{\sim}{\rightarrowtail} RY \twoheadrightarrow *$ and take the composite of functors R and Q.

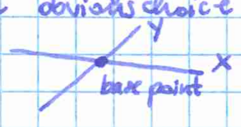
Why weakly equivalence?
$X \xrightarrow{\sim} X'$, X' cofibrant fibrant, so we take D as such?
[suppose we have w.e. here

2.4.2019 CYLINDER OBJECT AND LEFT HOMOTOPY IN TOP^*
pointed top. spaces

i) Coproduct?
Recall, that Top^* is pointed, and we'll use $*$ as the null-object. ($*$ both initial and final object as 1-pointed space)

$$\begin{tikzcd} * \arrow[r, "y_0"] \arrow[d, "x_0"] & Y \arrow[d] \arrow[ddr, bend left] & \\ X \arrow[r] \arrow[drr, bend right] & ? \arrow[dr, dashed] & \\ & & Z \end{tikzcd}$$

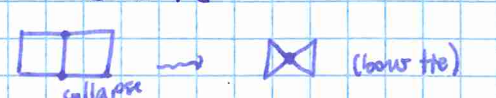
$X \sqcup Y/_{x_0 \sim y_0} =: X \vee Y$
↳ when we understand what are the base points
(it cannot be the disjoint union, because we don't have an obvious choice of a base point)



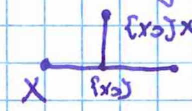
ii) cylinder $X \sqcup X$ free space is essentially the same as $X \vee X$ ← pointed space

$$\begin{tikzcd} X \vee X \arrow[d, "i"] \arrow[dr, "\nabla = id + id"] & \\ ? \; X \wedge I \arrow[r, "p"] & X \end{tikzcd}$$

, where $X \wedge I = X \times I /_{\{x_0\} \times I}$
$I = [0,1]$



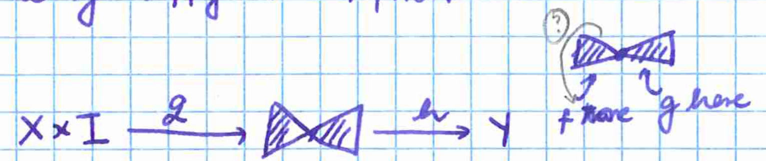
If X is cofibrant in Top^* ($\{x_0\} \to X$ is a closed fibration), then the collapsing map $p: X \times [0,1] \to X \wedge I$ is a homotopy equivalence.
Since $pr_X : X \times [0,1] \to X$ is a homotopy equivalence, so is p (check!)
Moreover, i is a cofibration (check!)
(if X is not cofibrant, we have to do a "cofibrant" replacement, e.g.



)

iii) Left homotopy from f to g $\forall f,g: X \to Y$, w.r.t. $X \wedge I$?

$$\begin{tikzcd} X \vee X \arrow[d, "i"] \arrow[dr, "f+g"] & \\ X \wedge I \arrow[r, "h"] & Y \end{tikzcd}$$

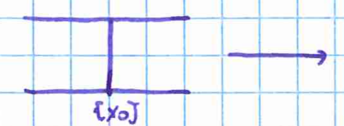


$hq : X \times [0,1] \to Y$
$hq(x_0, t) = y_0$ for all t.

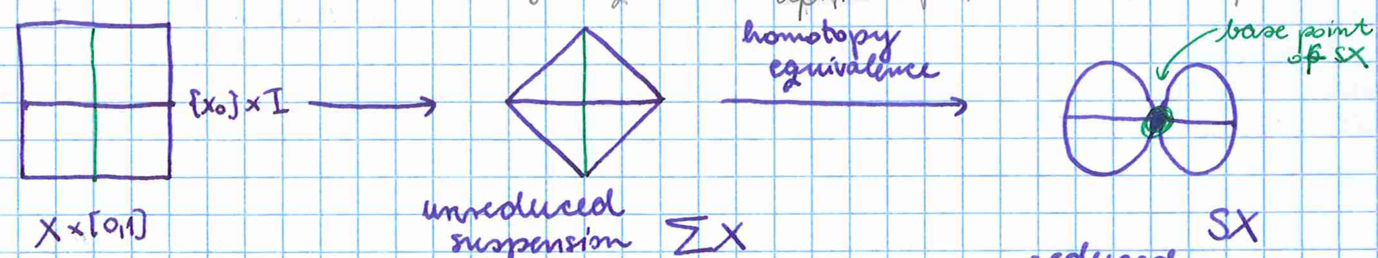
we call hq a base-point preserving homotopy $f \to g$ (obviously every b.-p. preserving homotopy $X \times I \to Y$ will induce a left homotopy $\hat{h}: X \wedge I \to Y$)
[bijective correspondence

iv) Suppose we're interested in left homotopies between constant maps! $X \xrightarrow{const\, y_0} Y$

They're in bijective correspondence with homotopies $X \times I \to Y$ that send $X \times \{0,1\} \cup \{x_0\} \times I$ to $\{y_0\}$.



in turn, these are in bijective correspondence with pointed maps $X \times I /_{(X \times \{0,1\} \cup \{x_0\} \times I)} \to Y$ ... smashed product
send all this to y_0 (precomposed with quotient map from before) what exactly is a smashed product?



(if collaps a ___, then get a ___ that is a htpy equivalence) in the ambient space,

Exercise. For $X = S^n$ (some base point), $n \geq 0$ the reduced suspension SS^n is homeomorphic to S^{n+1}!

Summary. Left homotopies $const_{y_0} \to const_{y_0}$ are in bijective correspondence with pointed continuous maps $SX \to Y$.

PATH OBJECTS, RIGHT HOMOTOPIES IN $\underline{Top^*}$

i) product

$$\begin{tikzcd} X\times Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & * \end{tikzcd}$$

$X\times Y$ has the obvious base-point (x_0, y_0)
(pointed maps to final object)

ii) path object

$$\begin{tikzcd} & \boxed{Y^I} \arrow[d] \\ Y \arrow[ur, "\sim"] \arrow[r] & Y\times Y \\ y_0 \arrow[r, mapsto] & (y_0,y_0) \end{tikzcd}$$

$\leftarrow$ choose const y_0 as base-point

iii) right-homotopy

Recall: every object is fibrant

$$\begin{tikzcd} & Y^I \arrow[d, "\varepsilon"] \\ X \arrow[ur, "k"] \arrow[r, "{(f,g)}"'] & Y\times Y \end{tikzcd}$$

$\varepsilon_0 k = f \quad \varepsilon_1 k = g$
$\varepsilon = (\varepsilon_0, \varepsilon_1)$

By adjunction to k there corresponds a map

$X\times[0,1] \longrightarrow Y$
$(x,t) \longmapsto (k(x))(t)$
$(x,0) \longmapsto f(x)$
$(x,1) \longmapsto g(x)$
$(x_0,t) \longmapsto y_0 \quad \forall t\in[0,1]$ right homotopy in Top^*, i.e. a pointed map

We see, that for each right homotopy, there corresponds a left homotopy.
not just adjunction, a bit of modding out

(iv) loop space

What are right homotopies between constant maps?
This gives $\varepsilon_0 k = \text{const}_{y_0}$, $\varepsilon_1 k = \text{const}_{y_0}$.

$$\begin{tikzcd} & \text{fibre of } \varepsilon \arrow[d] \\ X \arrow[ur, "k"] \arrow[r, "{(\text{const}_{y_0},\, \text{const}_{y_0})}"'] & \{(y_0,y_0)\} \end{tikzcd}$$

~~Recall~~ we see, that k maps into the loop space $\Omega Y = \Omega(Y, y_0)$.

Recall: left homotopies $\text{const}_{y_0} \Rightarrow$ are in bijective correspondence with constant maps $SX \to Y$
Right homotopies $\text{const}_{y_0} \Rightarrow$ are in bij. correspondence with ΩY^X.
(where X is cofibrant)

$(Y, y_0)^{(SX,*)} \longleftrightarrow (\Omega Y, \text{const}_{y_0})^{(X,x_0)}$

So we use "exponential law" to get a correspondence with maps $S^1 \to (Y,y_0)^{(X,x_0)}$

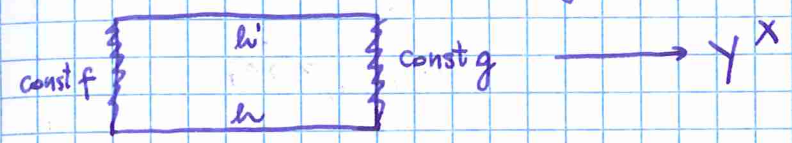
$[\underbrace{S\ldots(S(S(SX))\ldots)}_{n\text{-times}}, Y]_* = \pi_n\left((Y,y_0)^{(X,x_0)}, \text{const}_{y_0}\right)$

LEFT HOMOTOPY OF LEFT HOMOTOPIES IN ABSTRACT $\mathcal{C}$

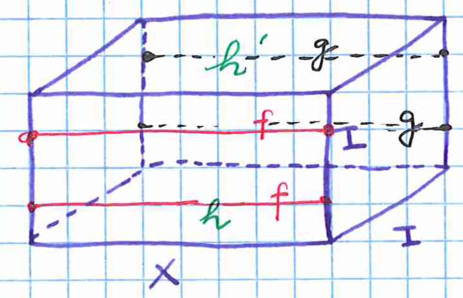
(not necessarily pointed)
↳ may have different initial and final objects

Motivation: cylinder in Top: $\_\_\_\_ \times [0,1]$
left homotopies from f to g: maps $X\times[0,1] \to Y$ $(\ldots f$ and $g)$
$\updownarrow$ correspond to paths from f to g in the function space $[0,1] \to Y^X$

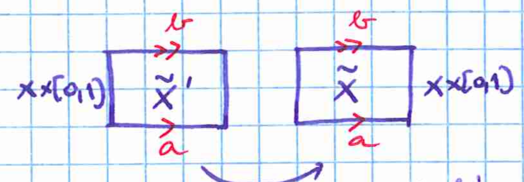
The natural relation to consider is homotopies between paths $[0,1] \to Y^X$ from f to g with fixed endpoints



Such a homotopy corresponds to a map $\underbrace{X\times[0,1]\times[0,1]}_{\approx} \longrightarrow Y$



$X\times[0,1]\times[0,1] / {\scriptstyle (x,0,t)\sim(x,0,0) \atop (x,1,t)\sim(x,1,0)} \rightleftarrows \tilde{\tilde{X}}$ collapse front of cube to a single copy of X and back to a single copy of X'



Let X be cofibrant and let Y be cofibrant (standing condition)
Let $h: \tilde{X} \to Y$ and $h': \tilde{X}' \to Y$ be homotopies from f to g.
An iterated cylinder object (for left homotopies of left homotopies) is a diagram

$$\begin{tikzcd} \tilde{X} \sqcup_{X\sqcup X} \tilde{X}' \arrow[d, "\iota+\iota'"'] \arrow[dr, "p+p'"] & \\ \tilde{\tilde{X}} \arrow[r, "\tilde{p}"'] & X \end{tikzcd}$$

iota

cylinder objects

$$\begin{tikzcd} X\sqcup X \arrow[d, "i"'] \arrow[dr, "\nabla"] & \\ \tilde{X} \arrow[r, "p"'] & X \end{tikzcd} \qquad \begin{tikzcd} X\sqcup X \arrow[d, "i'"'] \arrow[dr, "\nabla"] & \\ \tilde{X}' \arrow[r, "p'"'] & X \end{tikzcd}$$

($\tilde{X} = \tilde{X}'$ and we are just pretending they're different)

$$\begin{tikzcd} X\sqcup X \arrow[r, "i'"] \arrow[d, "i"'] & \tilde{X}' \arrow[d] \arrow[ddr, "p'"] \arrow[dddrr, bend left, "h'"] & & \\ \tilde{X} \arrow[r] \arrow[drr, bend right, "p"'] \arrow[ddrrr, bend right, "h"'] & \tilde{X}\sqcup_{X\sqcup X}\tilde{X}' \arrow[dr, "p+p'"] \arrow[ddrr, "h+h'"] & & \\ & & X & \\ & & & Y \end{tikzcd}$$

Def: A left homotopy of homotopies $h \Rightarrow h'$ w.r.t. $\tilde{\tilde{X}}$ is a commutative diagram

$$\begin{tikzcd} \tilde{X}\sqcup_{X\sqcup X}\tilde{X}' \arrow[d] \arrow[dr, "h+h'"] & \\ \tilde{\tilde{X}} \arrow[r] & Y \end{tikzcd}$$

CORRESPONDENCE BETWEEN LEFT AND RIGHT HOMOTOPIES FROM f TO g: $X \longrightarrow Y$ WHERE X IS COFIBRANT AND Y IS FIBRANT

Suppose we have an abstract cylinder object

$$
\begin{tikzcd}
X \sqcup X \arrow[d, "i"'] \arrow[dr, "\nabla"] & \\
\tilde{X} \arrow[r, "p"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \sqcup X \arrow[d, "i"'] \arrow[dr, "f+g"] & \\
\tilde{X} \arrow[r, "h"'] & Y
\end{tikzcd}
$$

path object

$$
\begin{tikzcd}
 & \hat{Y} \arrow[d, "\varepsilon"] \\
Y \arrow[ur, "c"] \arrow[r, "\Delta"'] & Y \times Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \hat{Y} \arrow[d] \\
X \arrow[ur, "k"] \arrow[r, "(f,g)"'] & Y \times Y
\end{tikzcd}
$$

Suppose given h and a fixed $\hat{Y}$.

(trivial cofib. $\Rightarrow$ so we have left...?)

$$
\begin{tikzcd}
 & X \arrow[r, "cg"] \arrow[d, "i_1"', "\sim"] & \hat{Y} \arrow[d, "\varepsilon"] \\
X \arrow[r, "i_0"] & \tilde{X} \arrow[r, "{(h, gp)}"'] \arrow[ur, "T"] & Y \times Y
\end{tikzcd}
$$

$k = T \circ i_0$
$h \circ i_1 = g$

Suppose given k and fixed $\tilde{X}$. By duality

$$
\begin{tikzcd}
X \sqcup X \arrow[r, "k+cg"] \arrow[d] & \hat{Y} \arrow[r, "\varepsilon_0"] \arrow[d, "\varepsilon_1", "\sim"'] & Y \\
X \arrow[r, "gp"'] \arrow[ur, "T"] & Y &
\end{tikzcd}
$$

(trivial, because Y is fibrant)

$p \circ i = \text{fold}$ $\quad g \circ \text{fold} = g+g$

$h = \varepsilon_0 T$ left homotopy

$$
\begin{tikzcd}
\cdot \arrow[r, dash, "\varepsilon_1 T = gp"] \arrow[d, dash, "Ti_0 = k"'] & \cdot \arrow[d, dash, "Ti_1 = cg"] \\
\cdot \arrow[r, dash, "\varepsilon_0 T = h"'] \arrow[ur, phantom, "T"] & \cdot
\end{tikzcd}
$$

↗ Quillen might just call it a lemma

Proposition. Suppose $T : \tilde{X} \longrightarrow \hat{Y}$ is a correspondence between left homotopy $h : \tilde{X} \longrightarrow Y$ and the right homotopy $k : X \longrightarrow \hat{Y}$ (from f to g), let $h' : \tilde{X}' \longrightarrow Y$ be another left homotopy from f to g. Then there is a correspondence $(\tilde{X}' \longrightarrow \hat{Y})$ ~~from~~ between h' and k iff h and h' are left homotopic.

from h to h'
(is reflexive)

Corollary. The relation of left homotopy is an equivalence relation on left homotopies.

Transitivity?

h, h', h'' left homotopies; T correspondence between h and k
$\Rightarrow \exists\, T'$ correspondence between h' and k

h left homotopic to h'
h' left homotopic to h''

$\Rightarrow \exists\, T''$ correspondence between h'' and k
$\Rightarrow$ ~~$\exists$~~ h is left homotopic to h'' $\square$

composition

$$
\begin{array}{c}
f_3 \\
\boxed{\; h' \;} \\
f_2 \\
\boxed{\; h \;} \\
f_1
\end{array}
$$

9.4.2019

Recall: correspondence between left and right homotopies

Fix cylinder object, fixed path object, X cofibrant, Y fibrant

$$
\begin{tikzcd}
X \sqcup X \arrow[d, "i"'] \arrow[dr, "\nabla"] & \\
\tilde{X} \arrow[r, "p"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \hat{Y} \arrow[d, "\varepsilon"] \\
Y \arrow[ur, "c"] \arrow[r, "\Delta"'] & Y \times Y
\end{tikzcd}
$$

(constant section)

$$
\begin{tikzcd}
X \sqcup X \arrow[d, "i"'] \arrow[dr, "f+g"] & \\
X \arrow[r, "h"'] & Y
\end{tikzcd}
$$

left htpy h

$$
\begin{tikzcd}
 & \hat{Y} \arrow[d, "\varepsilon"] \\
X \arrow[ur, "k"] \arrow[r, "{(f,g)}"'] & Y \times Y
\end{tikzcd}
$$

right homotopy k

Correspondence between h and k is a morphism $T : \tilde{X} \longrightarrow \hat{Y}$

$$
\begin{tikzcd}
X \sqcup X \arrow[r, "k+cg"] \arrow[d, "i"'] & \hat{Y} \arrow[d, "\varepsilon"] \\
\tilde{X} \arrow[r, "{(h, gp)}"'] \arrow[ur, "T"] & Y \times Y
\end{tikzcd}
$$

$\varepsilon_0 T = h$
$\varepsilon_1 T = gp$ (constant left homotopy from g to g).

$T i_0 = k$
$T i_1 = cg$ (constant right homotopy)

In Top, homotopies from f to g:
left homotopy $h : X \times [0,1] \longrightarrow Y$
right homotopy $k : X \longrightarrow Y^{[0,1]}$

Intuitively, h ~~also~~ corresponds to $k(x)(t) = h(x, t)$ (adjoint correspondence)
The formal correspondence (abstract, according to the definition)

$$T : X \times [0,1] \longrightarrow Y^{[0,1]}$$

$$T(x, s) = \big(t \longmapsto h(x, (1-t)s + t \cdot 1)\big)$$

The associated map $[0,1] \times [0,1] \longrightarrow Y^X$
(if we fix first coordinate, we get a homotopy between k and cg, similarly if we fix the second coordinate as in the diagram)

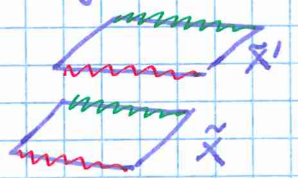
$$
\begin{tikzcd}
\cdot \arrow[r, dash, "gp"] \arrow[d, dash, "k"'] & \cdot \arrow[d, dash, "cg"] \\
\cdot \arrow[r, dash, "h"'] & \cdot
\end{tikzcd}
$$

Recall:

Proposition. Let h and k correspond and let h' be another left homotopy from f to g. Then h and h' are left homotopic iff h' and k correspond.

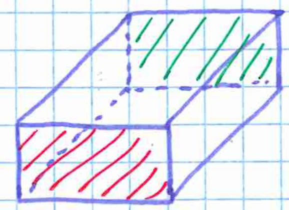
Proof. ($\Leftarrow$) Suppose h' and k correspond.
so we have a correspondence $T' : \tilde{X}' \longrightarrow \hat{Y}$.
We need to prove h and h' are left homotopic.
Fix an "iterated" cylinder object

$\tilde{X} \underset{X \sqcup X}{\sqcup} \tilde{X}'$



we glue over $X \sqcup X$.

$$\begin{tikzcd} \tilde{X} \underset{X\sqcup X}{\sqcup} \tilde{X}' \arrow[d, "\iota+\iota'"] \arrow[dr, "p+p'"] & \\ \tilde{\tilde{X}} \arrow[r, "\sim", "P\ (\text{upper case } P)"'] & X \end{tikzcd}$$



We are looking for

$$\begin{tikzcd} \tilde{X} \underset{X\sqcup X}{\sqcup} \tilde{X}' \arrow[d, "\iota+\iota'"] \arrow[dr, "h+h'"] & \\ \tilde{\tilde{X}} \arrow[r, "U"] & Y \end{tikzcd}$$

$$\begin{tikzcd} \tilde{X} \underset{X\sqcup X}{\sqcup} \tilde{X}' \arrow[d, "\iota+\iota'"'] \arrow[r, "T+T'"] & \hat{Y} \arrow[d, "\sim"', "\varepsilon_1"] \arrow[r, "\varepsilon_0"] & Y \\ \tilde{\tilde{X}} \arrow[r, "gP"] \arrow[ur, dashed, "V"] & Y & \end{tikzcd}$$

$\varepsilon_1 T = gp$
$\varepsilon_1 T' = gp'$

ε_1 is a trivial fibration
$\iota+\iota'$ is a cofibration (by definition) $\rightarrow$ we have a lifting V

We set $U = \varepsilon_0 V$
$U\iota = \varepsilon_0 V \iota = \varepsilon_0 T = h$
$U\iota' = \varepsilon_0 V \iota' = \varepsilon_0 T' = h'$

($\Rightarrow$) Suppose given U. Consider

$$\begin{tikzcd} & \tilde{X} \arrow[d, "\iota", "\sim"'] \arrow[r, "T"] & \hat{Y} \arrow[d, "\varepsilon"] \\ \tilde{X}' \arrow[r, "\iota'"] & \tilde{\tilde{X}} \arrow[r, "{(U, gP)}"] \arrow[ur, dashed, "V"] & Y \times Y \end{tikzcd}$$

$\varepsilon T = (h, gp)$
$U\iota = h$
$(gP)\iota = gp$

easy to prove ι is a trivial cofib (we by 2-out-of-3, it is a cofibration)
we have a lifting V
set $T' = V\iota'$
check $\varepsilon_0 T' = \varepsilon_0 V \iota' = U\iota' = h'$
$\varepsilon_1 T' = \varepsilon_1 V \iota' = gP\iota' = gp'$
$T' i_0' = V\iota' i_0' = V\iota i_0 = T i_0 = k$
$T' i_1' = V\iota' i_1' = V\iota i_1 = T i_1 = cg$

$$\begin{tikzcd} X \sqcup X \arrow[r, "i'"] \arrow[d, "i"] & \tilde{X}' \arrow[d] \arrow[ddr, bend left, "\iota'"] & \\ \tilde{X} \arrow[r] \arrow[drr, bend right, "\iota"] & \tilde{X} \underset{X\sqcup X}{\sqcup} \tilde{X}' \arrow[dr, "\iota+\iota'"] & \\ & & \tilde{\tilde{X}} \end{tikzcd}$$

so $\boxed{\iota' i' = \iota i}$

$\square$

Corollary. We can define a category $\mathrm{Hom}(X, Y)$ (X, Y fixed) with objects $f \in \mathrm{Hom}(X, Y) = \mathcal{C}(X, Y)$ and where a morphism $f \to g$ is an ~~class~~ equivalence class of left homotopies from f to g.
The set of equivalence classes is denoted by $\pi_1^{\ell}(X, Y; f, g)$.

[In Top, this would be ~~path~~ equivalence classes of paths into the function space $[0,1] \longrightarrow Y^{X}$ with resp. to homotopies fixing the end-points].

Compositions?

$f_1 \xrightarrow{h} f_2 \xrightarrow{h'} f_3$

$$\begin{tikzcd} X \sqcup X \arrow[d, "i"] \arrow[dr, "f_1+f_2"] & \\ \tilde{X} \arrow[r, "h"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \sqcup X \arrow[d, "i'"] \arrow[dr, "f_2+f_3"] & \\ \tilde{X}' \arrow[r, "h'"] & Y \end{tikzcd}$$

Recall how to construct an associated left homotopy $f_1 \to f_3$ from the proof of transitivity of left homotopy $\overset{\ell}{\simeq}$.

$$\begin{tikzcd} X \sqcup X \arrow[d, "j \circ i_0 + j' \circ i_1'"] \arrow[dr, "f_1+f_3"] & \\ \underset{X_{(1)} \equiv X'_{(0)}}{\tilde{X} \sqcup \tilde{X}'} \arrow[r, "h+h'"] & Y \end{tikzcd}$$

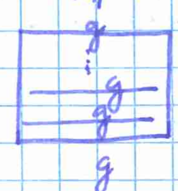
glue

Have to check this defines a composition
$\pi_1^{\ell}(X, Y; f_1, f_2) \times \pi_1^{\ell}(X, Y; f_2, f_3) \longrightarrow \pi_1^{\ell}(X, Y; f_1, f_3)$

The "inverse" of left homotopy h will give you an inverse map $g \to f$.

Identity morphisms?

$id \in \pi_1^{\ell}(X, Y; g, g)$

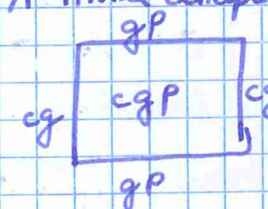


given by "constant left homotopy" gP with respect to some cylinder object (this can be any cyl. obj.)

Note, that

$$\begin{tikzcd} X \sqcup X \arrow[d, "i"] \arrow[r, "cg+cg"] & \hat{Y} \arrow[d, "\varepsilon"] \\ \tilde{X} \arrow[r, "{(gP, gP)}"] \arrow[ur, "cgP"] & Y \times Y \end{tikzcd}$$

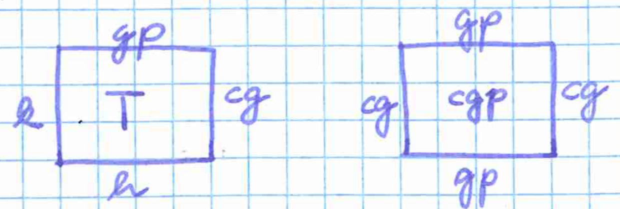
A trivial correspondence



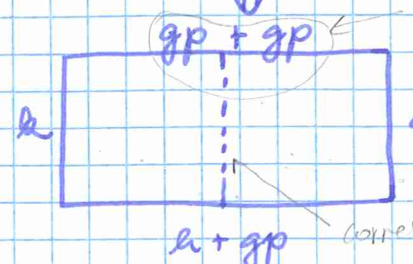
cg has a fixed path object, gP corresponds to cg for any cyl. object.
Since gP corresponds to cg, all gP (with respect to any cyl. object) are left homotopic.

Claim. $h + gp \simeq h$ ← doing this without correspondence is difficult, we need to redo the diagram chase from correspondence

Proof: Suppose h and k correspond (some right homotopy k, we pick a pair that corresponds)



⇓



this is gp for the glued mapping cylinder $\tilde{X} \sqcup \tilde{X}'$, $X_{(1)} \equiv X_{(2)}$

corresponding on $X_{(0)}$ and $X_{(1)}$ as cg

so this is a correspondence between $h+gp$ and k
$\Longrightarrow$ $h+gp$ and h are left homotopic
↑
by proposition cylinder:

$$\begin{tikzcd}
X \sqcup X \arrow[d, "j_0 i_0 + j_1 i_1"'] \arrow[drr, "\nabla"] & & \\
\tilde{X} \sqcup \tilde{X} \arrow[rr, "p+p"] & & X
\end{tikzcd}$$
$X_{(1)} \equiv X_{(0)}$

$j^0 : \tilde{X} \hookrightarrow \tilde{X} \sqcup \tilde{X}$ ($X_{(1)} \equiv X_{(0)}$) the inclusion map

$T_1\, cgp : \tilde{X} \to \hat{Y}$
$T + cgp : \tilde{X} \sqcup \tilde{X} \to \hat{Y}$ ($X_{(1)} \equiv X_{(0)}$)

$$\begin{tikzcd}
X \arrow[r, "i_0"] \arrow[d, "i_1"'] & \tilde{X} \arrow[d] \arrow[ddr, bend left, "cgp"] & \\
\tilde{X} \arrow[r] \arrow[drr, bend right, "T"'] & \tilde{X} \sqcup \tilde{X} \arrow[dr, "T+cgp"] & \\
& & \hat{Y}
\end{tikzcd}$$
$X_{(1)} \equiv X_{(0)}$

we just need to check the identities that come from the above rectangle. It is well defined on the glueing because T and cgp work well there.

Assume $\mathcal{C}$ is POINTED and X is a fixed cofibrant, Y fibrant.
Have a distinguished morphism
$0 : X \to * \to Y$ (composition, the null map, because it factors through the initial-final object $*$)
(analogue of the constant map)
Then $\pi_1^\ell(X, Y; 0, 0) =: \pi_1(X,Y)$ is a group (ℓ is also gone, because for right homotopies it's a bijection)
[In $\underline{Top^*}$, $\pi_1((Y,*)^{(X,*)}, const_*)$]

Consider a fixed cylinder object
$$\begin{tikzcd}
X \sqcup X \arrow[d, "i"'] \arrow[dr, "\nabla"] & \\
\tilde{X} \arrow[r, "p"] & X
\end{tikzcd}$$

and the pushout diagram
$$\begin{tikzcd}
X \sqcup X \arrow[r] \arrow[d, "i"'] & * \arrow[d] \\
\tilde{X} \arrow[r, "q"] & \Sigma X
\end{tikzcd}$$
ΣX: abstract suspension
(it is cofibrant, because it is a pushout of a cofibration)

[in $\underline{Top^*}$
$$\begin{tikzcd}
X \vee X \arrow[r] \arrow[d] & * \arrow[d] \\
X \times [0,1] / \{x_0\} \times [0,1] \arrow[r] & SX
\end{tikzcd}$$
]
SX: this is the reduced suspension in $\underline{Top^*}$

Theorem. The map $[\Sigma X, Y] \longrightarrow \pi_1^\ell(X,Y)$ ← left-homotopy classes
(In $\underline{Top^*}$, $[SX, Y]_* \cong \pi_1((Y,*)^{(X,*)}, *)$)
defined by $[\phi] \longmapsto [\phi \circ q]$
as Y ranges through fibrant objects of $\mathcal{C}$, is a natural equivalence of functors on $\mathcal{C}_f$ (with range $\underline{Grp}$).
↑ $\mathcal{C}$ on the fibrant objects

Σ induces a functor on the Homotopy category $Ho\,\mathcal{C} \to Ho\,\mathcal{C}$.
(we have an outside cylinder object $\tilde{X}$)

Dual. $[X, \Omega Y] \longrightarrow \pi_1^r(X,Y)$,
Ω induces a functor on the $Ho\,\mathcal{C} \to Ho\,\mathcal{C}$
Adjoint pair on $Ho\,\mathcal{C}$
$[\Sigma X, Y] \cong [X, \Omega Y]$

Remark.
If X cofibrant, then
$$X \xrightarrow[\sim]{i_s} \tilde{X}$$
it is a trivial cofibration (and we need this for the lifting property)
Similarly why we need Y to be fibrant.