Higher inductive types

① Recall:

→ property vs. structure

↪ $f : X \to Y$

$$\mathrm{isiso}(f) := \sum (g : Y \to X).\ f \circ g = id_Y \times g \circ f = id_X$$

→ equivalence

$$\mathrm{isequiv}(f) := \prod (y : Y).\ \mathrm{iscontr}(\underbrace{\mathrm{hfib}(f, y)})$$

$$\mathrm{hfib}(f, y) := \sum (x : X).\ fx =_Y y$$

$\mathrm{isprop}(\mathrm{isequiv}(f))$

$$X \simeq Y \quad := \quad \sum (f : X \to Y).\ \mathrm{isequiv}(f)$$

→ bi-invertible

$$\mathrm{isbiinv}(f) := \left(\sum (g : Y \to X).\ f \circ g = id_Y\right) \times \left(\sum (h : Y \to X).\ h \circ f = id_Y\right)$$

$\mathrm{isprop}(\mathrm{isbiinv}(f))$ ✓

$$\mathrm{isbiinv}(f) \simeq \mathrm{isequiv}(f)$$

→ half-adjoint equivalence (HoTT book)

Example: Homotopy pullbacks

Usual pullback:

$$
\begin{tikzcd}
Q \arrow[dr, dashed, "r"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\
& P \arrow[r, "p_1"'] \arrow[d, "p_2"] \arrow[dr, phantom, "=\alpha"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}
$$

(P, p_1, p_2, α) such that

$\forall$ such (Q, u, v, β), $\exists! r.$ ......

$\prod$ such (Q, v, u, β). iscontr(.....)
?

Homotopy pullback: given $f : X \to Z$
$g : Y \to Z$

$$
\begin{tikzcd}
Q \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "="] & {} \arrow[d, "f"] \\
{} \arrow[r, "g"'] & {}
\end{tikzcd}
$$

$$Sq(f,g) := \sum (Q : \mathcal{U})(u : Q \to X)(v : Q \to Y) .\ f \circ u = g \circ v$$

$$
\begin{tikzcd}
R \arrow[dr, "r"] \arrow[drr, bend left, "u \circ r"] \arrow[ddr, bend right, "v \circ r"'] & & \\
& Q \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "=\beta"] & {} \arrow[d, "g"] \\
& {} \arrow[r, "f"'] & {}
\end{tikzcd}
\qquad (*)
$$

whisker: $\prod (R : \mathcal{U})((Q, u, v, \beta) : Sq(f,g))(r : R \to Q) .\ Sq(f,g)$

whisker$(R, Q, u, v, \beta, r) := (R, u \circ r, v \circ r, \beta')$

$\hookrightarrow$ β': patch together $(*)$

$(P, p_1, p_2, \alpha) : Sq(f,g)$ is a pullback when the map

whisker$(R, P, p_1, p_2, \alpha, -) : (R \to P) \longrightarrow Sq(f,g)$

is an equivalence, for all R.

② Inductive types

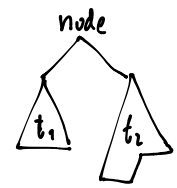
(General theory : W-types, well-founded trees)

Trees (binary) :

- Tree type "binary trees"
- empty : Tree
- node : Tree $\times$ Tree $\to$ Tree

empty

node(node(empty, empty), empty)



- Induction principle :

  Given : $P : \text{Tree} \to \mathcal{U}$

$$\text{ind}^P_{\text{Tree}} : P\,\text{empty} \times \left(\prod(t_1, t_2 : \text{Tree}).\ P t_1 \times P t_2 \to P(\text{node}(t_1, t_2))\right) \to \prod(t : \text{Tree}).P t$$

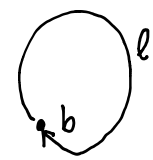
- equations :

$$\text{ind}^P_{\text{Tree}}(u, f, \text{empty}) \equiv u$$

$$\text{ind}^P_{\text{Tree}}(u, f, \text{node}(t_1, t_2)) \equiv f(t_1, t_2, \text{ind}^P_{\text{Tree}}(u, f, t_1), \text{ind}^P_{\text{Tree}}(u, f, t_2))$$

③ Higher inductive types

Example: the circle



- S^1 type "the circle"
- point constructors:

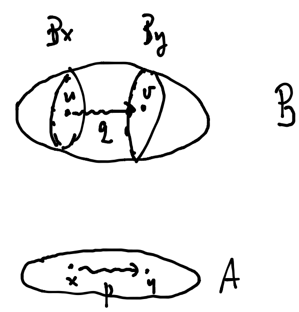
  $b : S^1$ "the base point"

- path constructors:

  $\ell : b =_{S^1} b$ "the loop"

- induction principle?

Preparation:



1) Path over path:

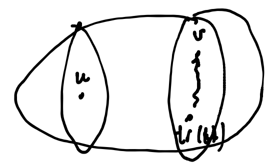
have: $B : A \to \mathcal{U}$

$x, y : A$, $u : Bx$

$v : By$

$p : x =_A y$

Want the space of "paths over p from u to v":

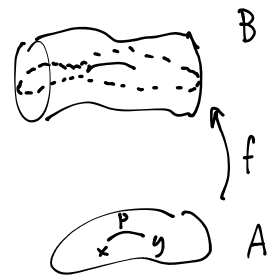


$$u =^{B}_{p} v \quad := \quad \mathrm{transport}^{B}(p, u) =_{By} v$$

$$\simeq \quad u =_{Bx} \mathrm{transport}^{B}(p^{-1}, v)$$

2) Dependent application:

$$f : \prod(x:A)\, B(x)$$



Given $p : x =_A y$

Want $f(p) : fx =^B_p fy$

$$\prod(x,y:A)(p : x =_A y).\ fx =^B_p fy$$

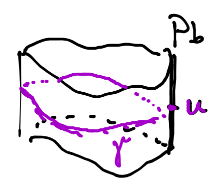
Path induction!

$$\prod(z:A).\ fz =^B_{\mathrm{refl}_z} fz$$

$$\lambda z .\ \mathrm{refl}_{fz}$$

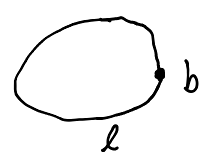
$$\underline{\quad\quad} . \underline{\quad\quad}$$

Induction principle for S^1:



$$P : S^1 \to \mathcal{U}$$

$$\mathrm{ind}^P : \prod(u:Pb).\ u =^P_\ell u \to \prod(z:S^1). Pz$$



Equations:

$$\mathrm{ind}^P(u,\gamma,b) \equiv u$$

$$\underbrace{\mathrm{ind}^P(u,\gamma,\ell)}_{\mathrm{apd}(\mathrm{ind}^P(u,\gamma,-))(\ell)} \equiv \gamma$$

Example: Propositional truncation

Given $A : \mathcal{U}$, we have:

$$A \xrightarrow{|\text{-}|} \|A\|_{-1}$$
$$a \longmapsto |a|$$

- $\|A\|_{-1}$ type

- point constructors:

$$|\ | : A \to \|A\|_{-1} \qquad \frac{a : A}{|a| : \|A\|_{-1}}$$

- path constructors:

$$\underline{p} : \prod(\alpha, \beta : \|A\|_{-1}).\ \alpha =_{\|A\|_{-1}} \beta$$

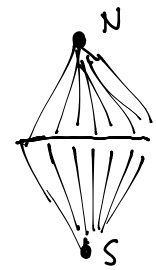
- induction principle:

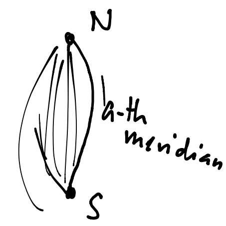
$$P : \|A\| \to \mathcal{U}$$

$$\begin{aligned} \mathrm{ind}^P : &\left(\prod(x : A).\ P(|x|)\right) \to \\ &\left(\prod(\alpha, \beta : \|A\|_{-1})(u : P\alpha)(v : P\beta).\ u =^P_{\underline{p}(\alpha,\beta)} v\right) \to \\ &\prod(\xi : \|A\|_{-1}).\ P\xi. \end{aligned}$$

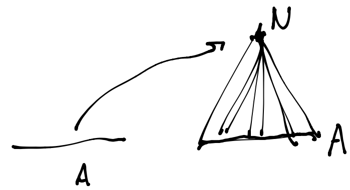
+ equations

Example: Suspension of A





$a : A$



- ΣA type
- point constructors:

  $N : \Sigma A$

  $S : \Sigma A$

- path constructors:

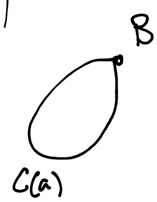
  $m : A \to N =_{\Sigma A} S$

- induction principle:

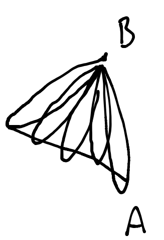
  $P : \Sigma A \to \mathcal{U}$

- "given a lifting of N, S and all the meridians, we get a section of P."

Alternative: (by Jahn)

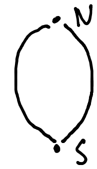


for $a : A$] to think about



$\Sigma 0 \simeq 2$

$\Sigma(\Sigma 0) \simeq \Sigma 2 \simeq S^1$



$\Sigma S^1 \simeq S^2$?

S^2 :

point: $b : S^2$ base

path: surf : $\mathrm{refl}_b \underset{b=b}{=} \mathrm{refl}\ b$