Equivalences

① Review:

homotopy levels

-2	:	contractible spaces	(exactly 1 point)
-1	:	propositions	(at most 1 point)
0	:	sets	(space whose path spaces are propositions)
1	:	groupoids	(space whose path spaces are sets)
$\vdots$			

up to paths

② Structure vs. property

Example: Groups

the type of groups:

$$\mathrm{Group} := \sum (G : \mathcal{U})(e : G)(m : G \times G \to G)(i : G \to G).\ \mathrm{isgroup}(G, e, m, i)$$

$$\mathrm{isgroup}(G, e, m, i) :=$$

$$\alpha : \Big(\prod (x, y, z : G).\ m(m(x,y),z) =_G m(x, m(y,z))\Big) \times$$

$$\Big(\prod (x : G).\ m(x,e) = x \times m(e,x) = x \times m(x, i(x)) = e \times m(i(x), x) = e$$

Write $m(x,y)$ as $x \cdot y$.

$$
\begin{tikzcd}
& ((x\cdot y)\cdot z)\cdot t \arrow[ddl, squiggly, "\alpha\ldots"'] \arrow[dr, squiggly, "\alpha\ldots"] & \\
& & (x(y\cdot z))\cdot t \arrow[d, squiggly, "\alpha\ldots"] \\
(x\cdot y)\cdot(z\cdot t) \arrow[dr, squiggly, "\alpha\ldots"'] & \overset{?}{=} & x\cdot((y\cdot z)\cdot t) \arrow[dl, squiggly, "\alpha\ldots"] \\
& x\cdot(y\cdot(z\cdot t)) &
\end{tikzcd}
$$

Wrong? Solution:

make sure that G is a set:

$$\mathrm{Group} := \sum (G : 0\text{-Type}) \ldots\ldots$$

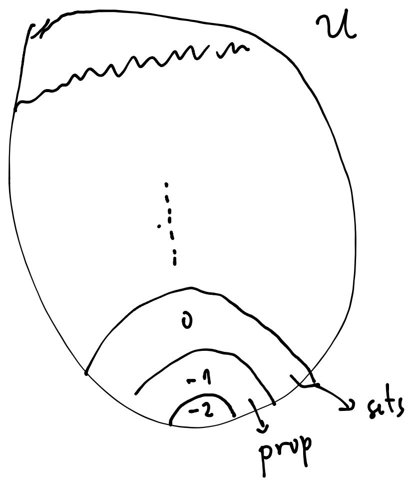
<u>Other examples:</u>

- algebraic structures (rings, modules) the carrier type should be a set.
- a category:

$\mathbb{C}_0 : \mathcal{U}$ space of objects

$\mathbb{C}_1 : \mathbb{C}_0 \times \mathbb{C}_0 \to 0\text{-Type}$ space of morphisms $A \to B$ $\mathbb{C}_1(A,B)$

$$0\text{-Type}_{\mathcal{U}} := \sum (X : \mathcal{U})\ \mathrm{isset}(X)$$

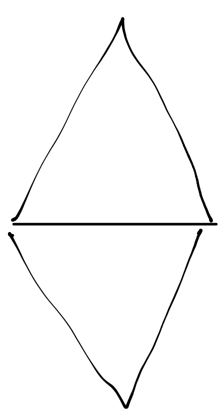


Example: Cauchy sequence? $a : \mathbb{N} \longrightarrow \mathbb{Q}$

$$\forall \varepsilon > 0 .\ \exists n \in \mathbb{N} .\ \forall m, k \geq n .\ |a_m - a_k| < \varepsilon$$

$$c : \prod \varepsilon > 0 .\ \sum (n : \mathbb{N}) .\ \prod m, k \geq n .\ |a_m - a_k| < \varepsilon$$

$$c(\varepsilon) = (\underset{\pi_1(c(\varepsilon))}{\underset{\shortparallel}{n_\varepsilon}}, \ \ldots)$$



~~Theorem ....~~

~~Proof .....~~

Problem : ....

Construction/Solution: ....

Property of (points of) A is a dependent type

$$P : A \longrightarrow \mathrm{Prop} \qquad \mathrm{Prop} := (-1)\text{-Type}$$

Structure on A is a dependent type

$$P : A \longrightarrow \mathcal{U}$$

Example:

Theorem: For a group homomorphism $f : G \to H$,

$$G/\mathrm{Ker}\, f \cong \mathrm{Im}\, f.$$

Structure because $A \cong B$ can have many elements

Theorem: For a group homomorphism $f : G \to H$, the canonical map $G/\mathrm{Ker}\, f \longrightarrow \mathrm{Im}\, f$ is an isomorphism.

property?

(3) Equivalences

Suppose $f : X \to Y$

$$\mathrm{isiso}(f) := \sum (g : Y \to X).\ (g \circ f =_{X\to X} id_X) \times (f \circ g =_{Y \to Y} id_Y)$$

Is isiso(f) a proposition?

$(g_1, \eta_1, \theta_1) : \mathrm{isiso}(f)$

$(g_2, \eta_2, \theta_2) : \mathrm{isiso}(f)$

$\eta_1 : g_1 \circ f = id_X$

$\vdots$

$\eta_2 : g_2 \circ f = id_X$

Is it the case that $g_1 \overset{?}{=} g_2$

What can be done (exercise):

$$\prod (y : Y).\ g_1 y = g_2 y \qquad (*)$$

Def: Homotopy between $u, v : A \to B$ is a point of

$$(u \sim v) := \prod (a : A).\ u\,a =_B v\,a$$

Is it the case that $g_1 \sim g_2 \to g_1 \overset{?}{=} g_2$

Function extensionality:

$$\mathrm{funext} : \prod (X, Y : \mathcal{U})(f, g : X \to Y).\ \left(\prod (x : X)\ f x =_Y g x\right) \to f =_{X \to Y} g$$

Assume there is a point funext.

Thus using funext we can show $p : g_1 = g_2$ (continue from $(*)$)

Still need to show $\mathrm{transport}(p, \eta_1) = \eta_2$ and similarly for θ

Cannot be done.

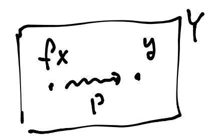
$\Rightarrow$ $\mathrm{isiso}(f)$ is not in general a proposition.

Need to improve to a proposition.

Def : For $f : X \to Y$ and $y : Y$, let

$$\mathrm{hfib}(f, y) := \sum (x : X).\ fx =_Y y$$

$$(x, p)$$



Def : For $f : X \to Y$, let

$$\mathrm{isequiv}(f) := \prod (y : Y).\ \mathrm{iscontr}(\mathrm{hfib}(f, y))$$

"All homotopy fibers of f are contractible (canonically/continuously in y)"

"The inverse image of every point is a singleton."

Theorem: $\mathrm{isprop}(\mathrm{isequiv}(f))$.

Define : $X \simeq Y$ defined to be $\sum (f : X \to Y).\ \mathrm{isequiv}(f)$.

$X \cong Y$ —"— $\sum (f : X \to Y).\ \mathrm{isiso}(f)$

Observations:

- $\mathrm{isequiv}(\mathrm{id}_X) \equiv \prod(y:X).\ \mathrm{iscontr}(\overbrace{\mathrm{hfib}(\mathrm{id}_X, y)}^{\Sigma(x:X).\, x =_X y})$

$$\equiv \prod(y:X).\ \underbrace{\mathrm{iscontr}\left(\Sigma(x:X).\ x =_X y\right)}$$

Center of contraction: $(y, \mathrm{refl}_X\, y)$

also: $\prod(z:X)(p: z = y).\ (z, p) = (y, \mathrm{refl}_X\, y)$

- inverse: $f : X \to Y$

$$c : \mathrm{isequiv}(f) \equiv \prod(y:Y).\ \mathrm{iscontr}(\mathrm{hfib}(f, y))$$

Take $y : Y$: $\pi_1(cy) : \mathrm{hfib}(f, y)$

$\pi_1(\pi_1(cy)) : X$

Have $f^{-1} : y \mapsto \pi_1(\pi_1(cy))$

$Y \longrightarrow X$

$\mathrm{inv}(f, c)$ better

It turns out that f^{-1} is an inverse of f, and also f^{-1} is an equivalence:

$$X \simeq X$$

$$X \simeq Y \to Y \simeq X$$

better: $(X \simeq Y) \simeq (Y \simeq X)$

$$X \simeq Y \times Y \simeq Z \longrightarrow X \simeq Z$$

Relationship between $X \simeq Y$ and $X \cong Y$:

$$X \simeq Y \longrightarrow X \cong Y$$

$$\mathrm{isequiv}(f) \longrightarrow \mathrm{isiso}(f)$$

Also: $$X \cong Y \longrightarrow X \simeq Y$$

<u>Theorem</u>:

(a) If P, Q are propositions then
$$(P \simeq Q) \simeq (P \to Q) \times (Q \to P)$$

(b) If X, Y are sets then
$$(X \simeq Y) \simeq (X \cong Y)$$