~~H-levels~~ n-Types

(1) Review:

Identity type

$$\left.\begin{array}{l} Id_A(x,y) \\ x =_A y \\ x = y \end{array}\right\}$$ The space of paths from x to y in A

Structure: $\left.\begin{array}{l} refl_A\, x \\ refl\, x \end{array}\right\}$ constant path at x

Path induction:

want $J(\dots) : \prod (x\ y : A)(p : x =_A y).\ B(x, y, p)$

suffices $d : \prod (z : A).\ B(z, z, refl_A\, z)$

$=_A$ "equivalence relation":

$refl\, x : \quad x =_A x$

inverse paths: $\quad p : x = y \quad \to \quad p^{-1} : y = x$

Concatenation: $\quad \begin{array}{l} p : x = y \\ q : y = z \end{array} \quad \to \quad p \cdot q : x = z$

$$\begin{tikzcd} x \arrow[r, "p"] \arrow[rr, bend right, "p \cdot q"'] & y \arrow[r, "q"] & z \end{tikzcd}$$

Such that: $\quad \prod (x\ y : A)(p : x = y).\ p^{-1} \cdot p = refl\, y$

$y =_A y$

$p \cdot p^{-1} = refl$

$(p \cdot q) \cdot r = p \cdot (q \cdot r)$

$p \cdot refl = p = refl \cdot p$

Mapping paths:

$$f : A \to B$$

$$p : x =_A y \qquad f \cdot p : fx =_B fy$$

"functorial" $\qquad f \cdot \mathrm{refl}_A x = \mathrm{refl}_B (fx)$

$$f \cdot (p \cdot q) = (f \cdot p) \cdot (f \cdot q)$$

(2) Contractible spaces

Def: A type A is contractible when (there is a point in) $\sum (x:A) \prod (y:A)\, x =_A y$.

$$\mathrm{iscontr}(A) :\equiv \sum (x:A). \prod (y:A)\, x =_A y.$$

Traditional reading:

"There is a point $x : A$ and a continuous map $\qquad (1)$

$$p : A \to A^{[0,1]}$$

such that $p(y)(0) = x$ and $p(y)(1) = y$."

Exercise: $(1) \iff A$ contractible

Example: $\mathrm{iscontr}(\mathbf{1})$

Construction: $(\ast, \underbrace{f}_{\prod (y:\mathbf{1}).\ \ast =_{\mathbf{1}} y})$ where $f(\ast) :\equiv \underbrace{\mathrm{refl}_{\mathbf{1}} \ast}_{\ast =_{\mathbf{1}} \ast}$

③ <u>h-levels</u> or <u>truncation levels</u>

$0, 1, 2, 3, \ldots$ $\qquad$ $-2, -1, 0, 1, \ldots$

n-type $\qquad$ for $n = -2, -1, 0, 1, 2, \ldots$

$$(-2)\text{-type}(A) :\equiv \mathrm{iscontr}(A)$$
$$(n+1)\text{-type}(A) :\equiv \prod(x,y:A).\ n\text{-type}(x =_A y)$$

<u>(-1)-types:</u>

A is a (-1)-type

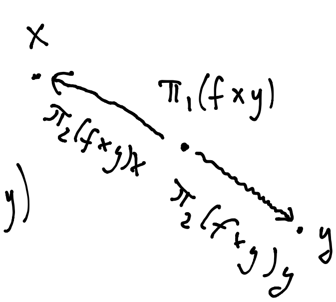
$$P :\equiv \prod(x,y:A).\ \mathrm{iscontr}(x =_A y)$$

Show that

$$P \rightleftarrows \prod(x,y:A)\ x =_A y$$

Suppose $f : P$. Want a point in

$$g : \prod(x,y:A)\ x =_A y$$



$$g :\equiv \lambda x\, y : A.\ (\pi_2(fxy)x)^{-1} \cdot (\pi_2(fxy)y)$$

Have $f\,x\,y : \mathrm{iscontr}(x =_A y)$

$$\pi_2(fxy) : \prod(z:A).\ \pi_1(fxy) =_A z$$

Suppose $g : \prod (x\, y : A)\ x =_A y$

Want $f : \prod (x\, y : A) . \sum (p : x =_A y) \prod (q : x =_A y) . p = q$

$$f = \lambda x\, y : A . \left(g\, x\, y \quad , \lambda q . \quad \underset{\text{HW.}}{} \quad \right)$$

(-1)-type (A) is just

$$\text{isprop}(A) :\equiv \prod (x\, y : A)\ x =_A y$$

Propositions (logical statements or truth values) can be represented as spaces.

Bad way: $\top$ is represented by $\mathbf{1}$
$\bot$ is represented by $\mathbf{0}$

Good way: $\cdot\ \{ u \in \mathbf{1} \mid p \}$ for p a truth value

Call the (-1)-types <u>propositions</u>.

0-Types (truncation level 0)

("UIP")

$$\prod(x\,y : A).\prod(p, q : x =_A y).\ p =_{x =_A y} q$$

Such a type has no "interesting" paths. We call it a set (think "all of its path-components are contractible")

$$\mathrm{isset}(A) :\equiv 0\text{-type}(A).$$

1-Types: groupoids

Observations

- $n\text{-type}(A) \to (n+1)\text{-type}(A)$
- not every type has a level: A $\quad n\text{-type}(A)$?

  example: S^2 (not explained yet)

④ Truncations

$$n\text{-Type} :\equiv \sum(A : \mathcal{U}).\ n\text{-type}(A)$$

$$\begin{array}{c} n\text{-Type} \\ \downarrow \uparrow ? \\ \mathcal{U} \end{array}$$

Def: The <u>n-truncation</u> of a type A is given by:

$\|A\|_n$ a type

$|-|_n : A \longrightarrow \|A\|_n$ i.e. if $a : A$ then $|a|_n : \|A\|_n$

Given $f : A \longrightarrow B$ where B is an n-type

we have $\bar{f} : \|A\|_n \longrightarrow B$ such that

$$\begin{tikzcd} A \arrow[r, "|\,|_n"] \arrow[dr, "f"'] & \|A\|_n \arrow[d, "\bar{f}"] \arrow[dl, phantom, "="] \\ & B \end{tikzcd}$$

$$\prod(x:A).\ f x =_B \bar{f}(|x|_n).$$

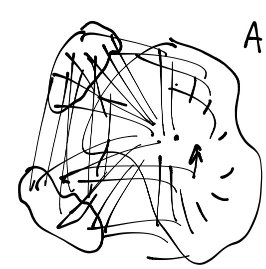
$\boxed{\text{NB: Actually need a dependent version where } f : \prod(x:A) B(x)}$

Propositional truncation

$\|A\|_{-1}$ traditionally corresponds to $A/\sim$ where $x \sim y$ for all $x, y : A$.

$\|0\|_{-1} \simeq 0$

if $a : A$ then $\|A\|_{-1}$ is contractible



A

Encoding of logic into type theory:

$$\top := 1$$
$$\bot := 0$$

Given P, Q : (-1)-types:

$$P \wedge Q := P \times Q$$
$$P \Rightarrow Q := P \to Q$$
$$\neg P := P \to 0$$
$$P \vee Q := \| P + Q \|_{-1}$$

Given $P : A \to (-1)\text{-Type}$:

$$\forall (x : A). P(x) := \prod (x : A). P(x)$$
$$\exists (x : A). P(x) := \| \sum (x : A). P(x) \|_{-1}$$

Intuitionistic logic. Can also assume:

$$\mathrm{lem} : \prod (P : (-1)\text{-Type}). P \vee \neg P.$$

Difference between $+$ and $\vee$, and Σ and $\exists$

$t : P + Q$ — t will tell us which one holds because it is of the form $\mathrm{inl}(-)$ or $\mathrm{inr}(-)$.

$\underbrace{t : \| P + Q \|_{-1}}_{}$ ($P \vee Q$) — t has form $| \mathrm{inl}(-) |_{-1}$ or $| \mathrm{inr}(-) |_{-1}$

Similarly: $t : \sum(x:A).P(x)$ concrete existence

$\pi_1 t : A$

$t : \|\sum(x:A).P(x)\|_{-1}$ abstract existence

Example:

$f : A \to B$

$$\left.\prod(y:B).\sum(x:A).\, fx =_B y \right\}$$ f is split epi (has a right inverse)

$$\left.\begin{array}{l}\prod(y:B).\|\sum(x:A).\, fx =_B y\|_{-1} \\ \forall(y:B).\ \exists(x:A).\, fx =_B (y)\end{array}\right\}$$ f is surjective

$$\mathrm{im}(f) := \sum(y:B).\ \exists(x:A).\, fx =_A y.$$

$\uparrow$
$\sum$ is wrong because then $\mathrm{im}(f) \simeq A$

0-truncation

$\pi_0(A) := \|A\|_0$ path-connected components of A

$a : A$ pointed type

$\Omega(A,a) := a =_A a$

$\pi_1(A,a) := \|\Omega(A,a)\|_0$

$\|A\|_1$ "fundamental groupoid"