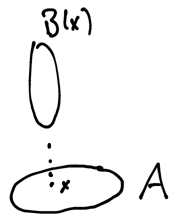
Recap: Type theory

type/space	A
points	$t : A$

dependent type $\quad x : A \vdash B(x)$ type

fibration $\quad B : A \to \mathcal{U}$

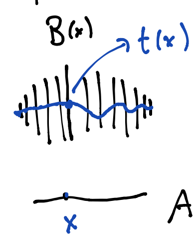
↳ universe of types



unit $\quad * : 1$

empty $\quad 0$

product $\quad \prod(x:A)\, B(x)$ $\quad$ space of sections



$\lambda(x:A).\, t(x)$

$x \mapsto t(x)$

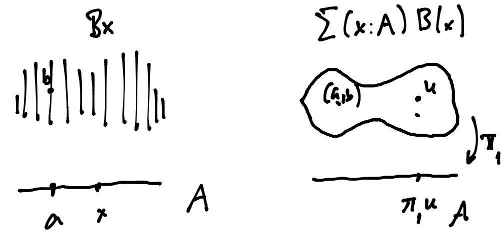
A map taking $x : A$ to some point $t(x) : B(x)$

$$\frac{f : \prod(x:A)\, B(x) \qquad a : A}{f\,a : B(a)}$$

function type $\quad A \to B := \prod(x:A)\, B$ or B^A

Sum
total space

$\sum (x:A)\, B(x)$



$(a,b) : \sum(x:A).\, B(x)$

where $a : A$,

$b : B(a)$

$\pi_1 : \sum(x:A)B(x) \to A$

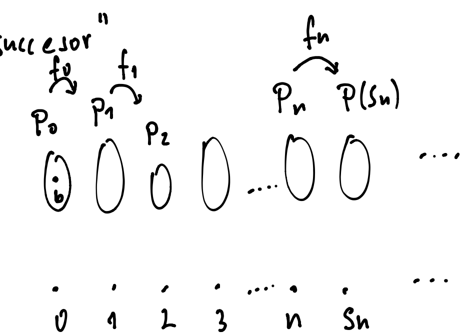
$\pi_2 : \prod\big(u : \sum(x:A)B(x)\big).\, B(\pi_1 u)$

Natural numbers $\mathbb{N}$

- $\mathbb{N} : \mathcal{U}$
- constructors

1) $0_{\mathbb{N}} : \mathbb{N}$

2) if $n:\mathbb{N}$ then $Sn:\mathbb{N}$ "succesor"



3) Induction principle:

We are given

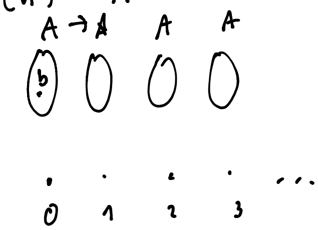
- $P : \mathbb{N} \to \mathcal{U}$
- $b : P(0)$
- $f : \prod(n:\mathbb{N}).\, P(n) \to P(Sn)$
- $k : \mathbb{N}$

Have: $\mathrm{ind}_{\mathbb{N}}(P, b, f, k) : P(k)$

Such that: $\mathrm{ind}_{\mathbb{N}}(P, b, f, 0) \equiv_{P(0)} b$

$\mathrm{ind}_{\mathbb{N}}(P, b, f, Sm) \equiv_{P(Sm)} f\, m\, (\mathrm{ind}_{\mathbb{N}}(P, b, f, m))$

Recursion principle: $A : \mathcal{U}$ $\qquad P(n) :\equiv A$



$b : A$

$f : \Pi(n : \mathbb{N}).\, A \to A \;\equiv$

$\qquad \mathbb{N} \to A \to A$

$k : \mathbb{N}$

Get: $\quad \mathrm{ind}_{\mathbb{N}}(\lambda\_.A, b, f, k) =: \mathrm{rec}_{\mathbb{N}}(A, b, f, k) : A$

Identity Types

In some Quillen model:

$$\begin{tikzcd}
A \arrow[r, "r_A", "\text{trivial cofibration}"'{yshift=30pt}] \arrow[d, "\Delta_A"] & P_A \arrow[dl, "\langle \varepsilon_0, \varepsilon_1 \rangle\ \text{fibration}"] \\
A \times A &
\end{tikzcd}$$

$P_A = A^{[0,1]}$

$r_A(x) = \lambda t.x$

$\qquad = \mathrm{const}_x$

$\varepsilon_0(\gamma) = \gamma(0)$

$\varepsilon_1(\gamma) = \gamma(1)$

fiber of P_A at $(a,b) \in A \times A$:

$\{\gamma : A^{[0,1]} \mid \gamma(0) = a,\ \gamma(1) = b\}$

Consider:

$$\begin{tikzcd}
A \arrow[rr, "d"] \arrow[d, "r_A"'] \arrow[dr, phantom, "="] & & B \arrow[d, "p\ \text{fibration}"] \arrow[dl, phantom, "="] \\
P_A \arrow[rr, "id"'] \arrow[urr, dashed, "J" description] \arrow[dr] & & P_A \arrow[dl, "\langle \varepsilon_0, \varepsilon_1 \rangle"] \\
& A \times A &
\end{tikzcd}$$

Identity type :

- formation rule: given $A : \mathcal{U}$, $a : A$, $b : A$ there is

$$\mathrm{Id}_A(a,b) : \mathcal{U}$$

Other notations: $a =_A b$ (not to confuse

$a = b$ with $a \equiv_A b$)

- Constructor: given $a : A$ there is

$$\mathrm{refl}_A(a) : \mathrm{Id}_A(a,a)$$

Other notations:

$$\mathrm{idpath}_A(a) : a =_A a$$

- path induction:

Given:

→ $B : \prod (x:A)(y:A)(p : x =_A y).\ \mathcal{U}$

For all $x : A$, $y : A$, $p : x =_A y$, have $B(x,y,p) : \mathcal{U}$.

→ $d : \prod (z:A).\ B(z, z, \mathrm{refl}_A(z))$

→ $a : A$, $b : A$, $q : a =_A b$

Have: $J(B, d, a, b, q) : B(a, b, q)$

$$J(B, d, c, c, \mathrm{refl}_A(c)) \equiv_{B(c,c,\mathrm{refl}\,c)} d(c) \quad \text{for } c : A$$

Path induction for humans:

- have $B(x, y, p)$ depends on $x, y : A$ and $p : x =_A y$

  NB: x, y and p must be unrestricted / arbitrary.

- Want point of $\prod (x, y : A)(p : x =_A y)\ B(x, y, p)$
- Sufficient to give a point of $\prod (z : A)\ B(z, z, \mathrm{refl}_A(z))$.

Example : Paths can be inverted

$$\mathrm{inv} : \prod (x, y : A).\ x =_A y \to y =_A x$$

Solution: $\prod (x : A).\prod (y : A).\prod (p : x =_A y).\ y =_A x$

Take $B(x, y, p) := y =_A x$

$$\mathrm{inv} : \prod (x, y : A)\ (p : x =_A y).\ B(x, y, p)$$

By path induction enough to give

$$d : \prod (z : A).\ B(z, z, \mathrm{refl}_A(z))$$

$$\prod (z : A).\ z =_A z$$

Take $d = \lambda(z : A).\ \mathrm{refl}_A(z)$.

Get $\underset{\substack{\uparrow \\ x =_A y}}{\mathrm{inv}(x, y, p)} : y =_A x$ Write $p^{-1} := \mathrm{inv}(x, y, p)$

Know: $\mathrm{refl}_A(z)^{-1} \equiv \mathrm{refl}_A(z)$.

Exercise: Paths can be composed

$$x \xrightarrow{p} y \xrightarrow{q} z$$

$$\prod (x,y:A)(p:x=y)\underbrace{(z:A)(q:y=z).\ x=z}_{B(x,y,p)}$$

Path induction:

$$\prod (v:A)\ B(v,v,\mathrm{refl}(v))$$

$$\mathrm{refl}_v \quad v \xrightarrow{q} z$$

$$d : \prod (v:A).\prod (z:A)(q:v=z).\ v=z$$

Take $d = \lambda v:A.\ \lambda z:A.\ \lambda q:v=z.\ q$

Write $p * q := J(B, d, x, y, p)(z)(q)$ for

$p : x=y,\ q : y=z$

Know: $\mathrm{refl}_A(x) * q \equiv_{x =_A z} q$ for $q : x = z$

Path composition is associative:

$$p * (q * r) =_{x =_A w} (p * q) * r$$

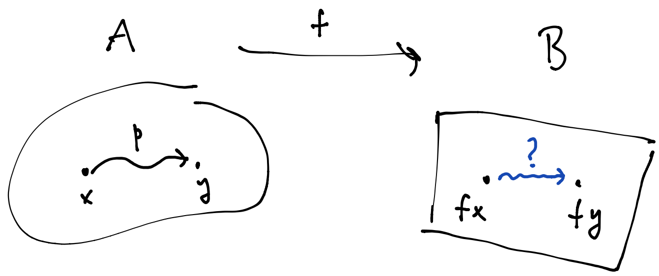
$$x \xrightarrow{p} y \xrightarrow{q} z \xrightarrow{r} w$$

Conclusion: $\sum (x:A).\sum (y:A).\ x =_A y$

(x, y, p) where $x : A$

$y : A$

$p : x =_A y$



$$\prod (x:A)(y:A)(p : x =_A y).\ \underbrace{fx =_B fy}_{B(x,y,p)}$$

Path induction

$$\prod (z:A).\ fz =_B fz$$

$$\lambda z.\ \mathrm{refl}_B(fz)$$

Notation: $\quad ap_f(p) : fx =_B fy$

$\qquad\qquad f(p)$

Know: $\quad f(\mathrm{refl}_z) \equiv \mathrm{refl}_B(fz)$

Exercises:

$$f(p^{-1}) \underset{fy =_B fx}{=} f(p)^{-1}$$

$$f(p * q) = f(p) * f(q)$$