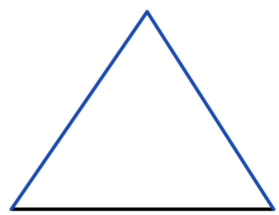
Type theory

a mathematical theory of constructions



Type .... describes a kind of construction

element or <u>a point</u> .... the resut of a construction

$$t : A$$

point t of type A ..... A reveals the structure of t

→ a type (of construction) A

→ the <u>extension</u> of a type A: the "collection" of points of A

- even primes larger than 3
- lines in the plane that are perpendicular to themselves

→ equations:

$s \equiv_A t$ points s and t of type A are equal

$A \equiv B$ types A and B are equal

→ dependency:

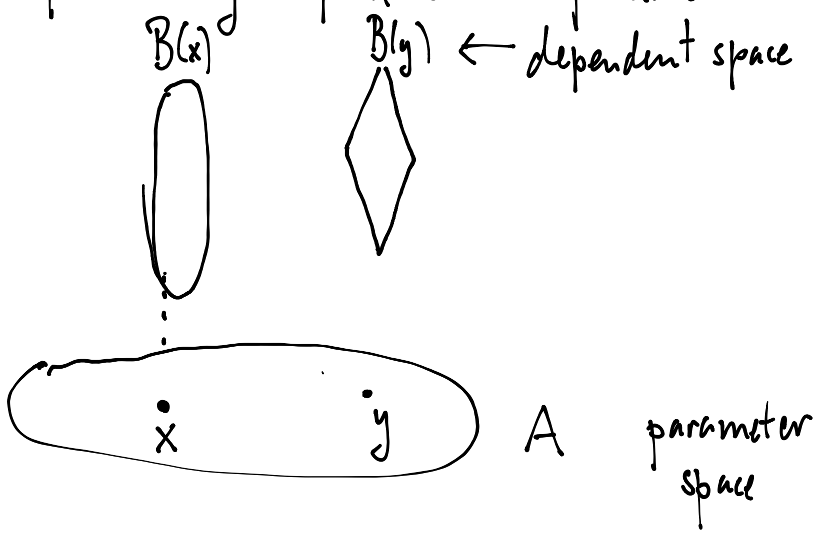
Consider $\mathcal{C}([a,b], \mathbb{R})$

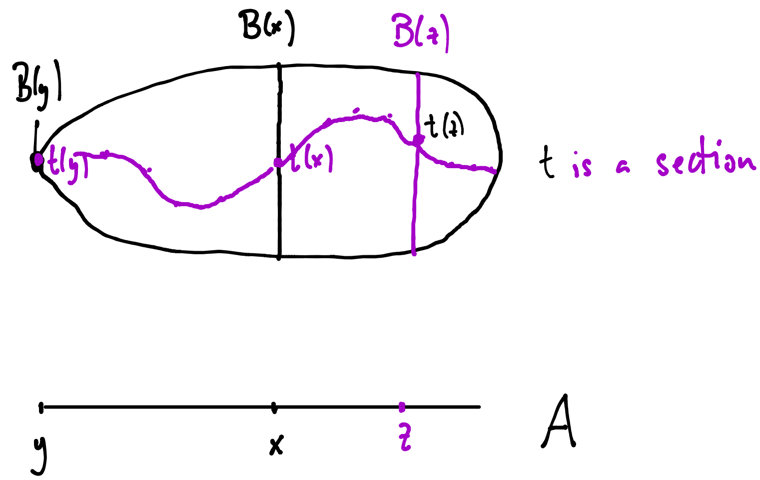
depends on $a, b : \underline{\underline{\mathbb{R}}}$

Two kinds of dependence:

→ a space may depend on a parameter

→ a point may depend on a parameter





$$\begin{array}{c} B \\ p \big\downarrow \\ A \end{array}$$

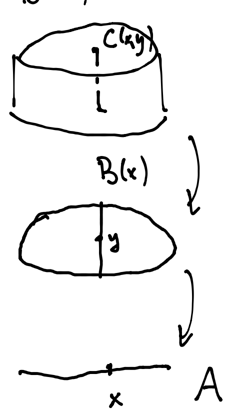
$x : A \vdash B(x)$ type

"$B(x)$ is a type which depends on the parameter $x : A$"

$$x_1 : A_1,\ x_2 : A_2(x_1),\ \dots,\ x_n : A_n(x_1, \dots, x_{n-1}) \vdash B(x_1, \dots, x_n) \text{ type}$$

$$x : A \vdash t(x) : B(x)$$

$x : A,\ y : B(x) \vdash C(x,y)$ type



$$\begin{array}{c} E \\ \big\downarrow \\ B \\ \big\downarrow \\ A \end{array}$$

Basic types: $\mathbb{0}, \mathbb{1}, \mathbb{2}, \mathbb{N}, \prod(x:A)B(x)$

$\sum(x:A)B(x)$

$\mathcal{U}$

New type:

- formation : how to form the new type
- constructors : how to construct points
- deconstructors : how to deconstruct/use points
- equations

The unit type $\mathbb{1}$ (one-point space)

$$\frac{P_1 \quad P_2 \quad \cdots \quad P_n}{C} \text{ rule}$$

premises

conclusion

- formation: $\dfrac{}{\mathbb{1} \text{ type}}$
- constructor: $\dfrac{}{* : \mathbb{1}}$
- deconstructor: /
- equations: $\dfrac{s:\mathbb{1} \quad t:\mathbb{1}}{s \equiv_{\mathbb{1}} t}$ alternative: $\dfrac{s:\mathbb{1}}{s \equiv_{\mathbb{1}} *}$

Rules for equality: $\dfrac{A \text{ type}}{A \equiv A}$ $\quad \dfrac{A \equiv B}{B \equiv A}$ $\quad \dfrac{A \equiv B \quad B \equiv C}{A \equiv C}$

The empty type 0

$$\frac{}{0 \text{ type}} \qquad \frac{e : 0 \qquad A \text{ type}}{\mathrm{absurd}_A(e) : A} \qquad \frac{e : 0 \quad s : A \quad t : A \quad A \text{ type}}{s \equiv_A t}$$

Exercise: Category $\mathcal{C}$, initial object 0, what is the slice $\mathcal{C}/0$?

$A \wedge B \Rightarrow A$

The dependent sum

Suppose type A and $B(x)$ depends on $x : A$.

Form dependent sum $\sum (x:A) . B(x)$ type.

$$\sum_{x:A} B(x)$$

Points of $\sum (x:A) . B(x)$ are pairs (s,t) where $s : A$ and $t : B(s)$.

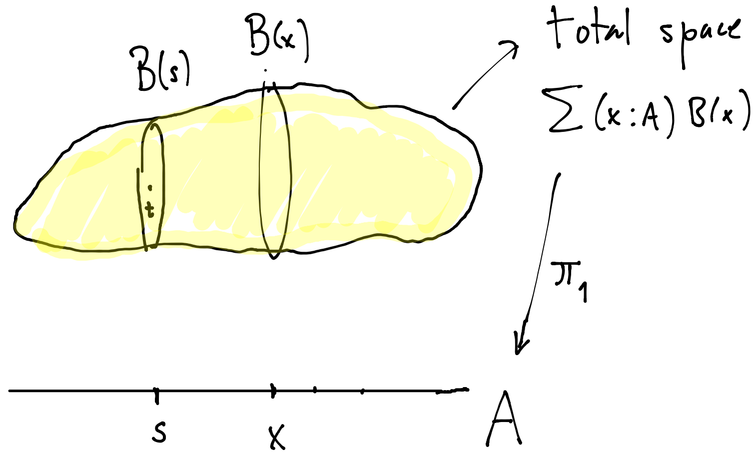
Given $p : \sum (x:A) . B(x)$ there are

$$\pi_1(p) : A$$
$$\pi_2(p) : B(\pi_1(p))$$

"projections"

Equations:

$$\pi_1(s,t) \equiv_A s \qquad (\pi_1 p, \pi_2 p) \equiv_{\sum(x:A).B(x)} p$$
$$\pi_2(s,t) \equiv_{B(s)} t$$



The dependent product

Given type A and $x:A \vdash B(x)$ type,

dependent product $\prod(x:A).B(x)$.

(The space of sections.)

If $x:A \vdash t(x):B(x)$ is a dependent point then

$$\underbrace{\lambda x:A.\,t(x)}_{x \mapsto t(x)} : \prod(x:A).B(x)$$

↳ ranges over A

Given $f:\prod(x:A).B(x)$ and $t:A$, have

$$f(t):B(t)$$

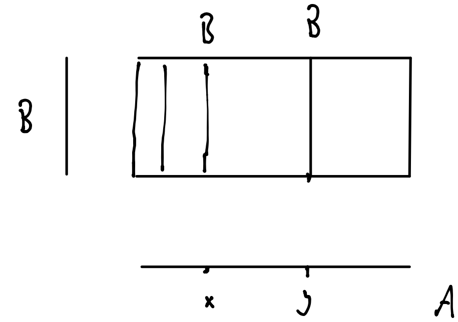
Equations: $(\lambda x:A.\,t(x))(s) \equiv_{B(s)} t(s)$ (replace x with s) where $x:A \vdash t(x):B(x)$

$$\lambda x : A . f(x) \equiv_{\Pi(x:A).B(x)} f \qquad \text{where } f : \Pi(x:A).B(x)$$

Special cases:

Suppose we have types A and B.

Define dependent type $C(x) := B$ for $x : A$



Then $\Sigma(x:A).C(x)$ is just the binary product of A and B:

$$A \times B := \Sigma(x:A).B$$

Similarly:

$$A \to B \text{ or } B^A := \Pi(x:A).B$$

maps from A to B.

Examples:

$$\lambda p . (\pi_2 p, \pi_1 p) \quad : A \times B \to B \times A$$

Universe: there is a type $\mathcal{U}$ which contains as its points "small types":

$$0,\ 1,\ 2,\ \mathbb{N},\ \sum(x:A).B(x)$$
$$\prod(x:A).B(x)$$
provided $A:\mathcal{U}$

$$x:A \vdash B(x):\mathcal{U}$$

The universe $\mathcal{U}$.

Can have a larger universe $\mathcal{U}:\mathcal{U}'$. Keep going $\mathcal{U}':\mathcal{U}'':\mathcal{U}'''\dots$

A dependent type $B(x)$ with parameter $x:A$ is just a point of $A \to \mathcal{U}$.

$$\begin{tikzcd} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathcal{U}} \arrow[d] \\ A \arrow[r] & \mathcal{U} \end{tikzcd}$$