1. Monade v matematiki

Omejimo se na množice

<u>Def</u>: <u>Monada</u> (T, η, μ) je podana z

- funktor $T : \underline{Set} \to \underline{Set}$:

$$\begin{aligned} T : Set &\to Set \\ A &\mapsto TA \end{aligned}$$

za $f : A \to B$, potem $Tf : TA \to TB$

$$T(id_A) = id_{TA}$$

$$A \xrightarrow{id_A} A$$

$$TA \longrightarrow TA$$

$$T(g \circ f) = Tg \circ Tf$$

- $\eta : Id_{set} \to T$ naravna transformacija

za $A \in Set$ imamo $\eta_A : A \to TA$

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & TA \arrow[d, "Tf"] \\ {} & {} \end{tikzcd}$$

$$Tf \circ \eta_A = \eta_B \circ f$$

MONADE

$\mu : T \circ T \to T$ za $A \in Set$ imamo $\mu_A : T(TA) \to TA$

naravna transf.

$$\begin{tikzcd}
T(TA) \arrow[r, "\mu_A"] \arrow[d, "T(Tf)"'] \arrow[dr, phantom, "="] & TA \arrow[d, "Tf"] & A \arrow[d, "f"] \\
T(TB) \arrow[r, "\mu_B"'] & TB & B
\end{tikzcd}$$

$A \xrightarrow{\eta_A} TA$

$TA \xrightarrow{T\eta_A} T(TA)$

$TA \xrightarrow{\eta_{TA}} T(TA)$

Zakoni:

$$\begin{tikzcd}
TA \arrow[r, "T\eta_A"] \arrow[dr, "id_{TA}"'] & T(TA) \arrow[d, "\mu_A"] & TA \arrow[l, "\eta_{TA}"'] \arrow[dl, "id_{TA}"] \\
{} \arrow[ur, phantom, "="] & TA &
\end{tikzcd}$$

$$\mu_A \circ T\eta_A = id_{TA} = \mu_A \circ \eta_{TA}$$

Asociativnost

$$\begin{tikzcd}
T(T(TA)) \arrow[r, "T\mu_A"] \arrow[d, "\mu_{TA}"'] \arrow[dr, phantom, "="] & T(TA) \arrow[d, "\mu_A"] \\
T(TA) \arrow[r, "\mu_A"'] & TA
\end{tikzcd}$$

$$\mu_A \circ \mu_{TA} = \mu_A \circ T\mu_A .$$

Primeri monad

1) Potenčna množica: $T = \mathcal{P}$

$A \longmapsto \mathcal{P}A$ potenčna množica

$f : A \to B$ $\quad\quad$ $\mathcal{P}f : \mathcal{P}A \longrightarrow \mathcal{P}B$

$$\begin{array}{c} S \\ \rotatebox[origin=c]{-90}{$\subseteq$} \\ A \end{array} \longmapsto \{f(x) \mid x \in S\} = f[S]$$

$$\eta_A : A \longrightarrow \mathcal{P}A$$
$$x \longmapsto \{x\}$$

$$\begin{tikzcd}
x \arrow[rrr, mapsto] \arrow[ddd, mapsto, bend right] & & & \{x\} \arrow[ddd, mapsto] \\
 & A \arrow[r, "\eta_A"] \arrow[d, "f"'] & \mathcal{P}A \arrow[d, "\mathcal{P}f"] & \\
 & B \arrow[r, "\eta_B"'] & \mathcal{P}B & \\
fx \arrow[rrr, mapsto] & & & \{fx\}
\end{tikzcd}$$

✓

$$(\mathcal{P}\,id_A)\, S = id_A[S] = S$$

✓

$$\mathcal{P}(g \circ f)\, S = (g \circ f)[S] = g[f[S]]$$

Potenčna množica

$$\mu_A = \bigcup$$

$$\mu_A : \mathcal{P}(\mathcal{P}A) \longrightarrow \mathcal{P}A$$

$$\mathcal{U} \longmapsto \bigcup \mathcal{U} = \{x \in A \mid \exists S \in \mathcal{U}.\, x \in S\}$$

$$\begin{tikzcd}
\mathcal{U} \arrow[rrr, mapsto, bend left] \arrow[ddd, mapsto, bend right] & & & \bigcup\mathcal{U} \arrow[dd, bend left] & \\
 & \mathcal{P}(\mathcal{P}A) \arrow[r, "\bigcup"] \arrow[d, "\mathcal{P}(\mathcal{P}f)"'] & \mathcal{P}A \arrow[d, "\mathcal{P}f"] & & \\
 & \mathcal{P}(\mathcal{P}B) \arrow[r, "\bigcup"'] & \mathcal{P}B & f[\bigcup\mathcal{U}] &)) ? \\
(\mathcal{P}f)[\mathcal{U}] \arrow[rrr, mapsto] & & & \bigcup(\mathcal{P}f)[\mathcal{U}] &
\end{tikzcd}$$

$$y \in f[\bigcup\mathcal{U}] \Leftrightarrow \exists x \in \bigcup\mathcal{U}.\, y = f(x) \Leftrightarrow \exists S \in \mathcal{U}.\, \exists x \in S.\, y = f(x)$$

$$y \in \bigcup(\mathcal{P}f)[\mathcal{U}] \Leftrightarrow \exists \underset{\overset{\parallel}{f[S]}}{R} \in (\mathcal{P}f)[\mathcal{U}].\, y \in R \Leftrightarrow \exists S \in \mathcal{U}.\, y \in f[S]$$

$$\Leftrightarrow \exists S \in \mathcal{U}.\, \exists x \in S.\, y = f(x).$$

Zakoni

$$\begin{tikzcd}
S \arrow[rr, mapsto] & & \{\{x\} \mid x \in S\} \arrow[dd, "\bigcup"] \\
\mathcal{P}A \arrow[r, "\mathcal{P}\eta_A"] \arrow[dr, swap, "id_{\mathcal{P}A}"] & \mathcal{P}(\mathcal{P}A) \arrow[d, "\mu_A"] & \\
& \mathcal{P}A & {}
\end{tikzcd}$$

$S \in \mathcal{P}A$

$$(\mathcal{P}\eta_A)S = \eta_A[S] = \{\{x\} \mid x \in S\}$$

$$\bigcup \{\{x\} \mid x \in S\} = S.$$

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}A) \arrow[d, swap, "\mu_A"] & \mathcal{P}A \arrow[l, swap, "\eta_{\mathcal{P}A}"] \arrow[dl, "id_{\mathcal{P}A}"] \\
\mathcal{P}A &
\end{tikzcd}$$

$\eta_{\mathcal{P}A}(S) = \{S\}$

$\mu_A(\eta_{\mathcal{P}A}(S)) = \bigcup\{S\} = S$

Asociativnost

$$\begin{tikzcd}
\mathcal{D} \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \mu_A[\mathcal{D}] \arrow[dd, mapsto] \\
& \mathcal{P}^3A \arrow[r, "\mathcal{P}\mu_A"] \arrow[d, "\mu_{\mathcal{P}A}"'] & \mathcal{P}^2A \arrow[d, "\mu_A"] & \\
& \mathcal{P}^2A \arrow[r, "\mu_A"'] & \mathcal{P}A & \bigcup\mu_A[\mathcal{D}] \arrow[dl, equal, "?"] \\
\bigcup\mathcal{D} \arrow[rr, mapsto] & & \bigcup(\bigcup\mathcal{D}) &
\end{tikzcd}$$

$$x \in \bigcup(\bigcup\mathcal{D}) \Leftrightarrow \exists S \in \bigcup\mathcal{D} \,.\, x \in S \Leftrightarrow \exists \mathcal{U} \in \mathcal{D} \,.\, \exists S \in \mathcal{U} \,.\, x \in S \,.$$

$$x \in \bigcup\mu_A[\mathcal{D}] \Leftrightarrow \exists \mathcal{U} \in \mathcal{D} \,.\, x \in \mu_A(\mathcal{U}) \Leftrightarrow \exists \mathcal{U} \in \mathcal{D} \,.\, x \in \bigcup\mathcal{U}$$

$$\Leftrightarrow \exists \mathcal{U} \in \mathcal{D} \,.\, \exists S \in \mathcal{U} \,.\, x \in S \,.$$

Maybe

$$TA = \underbrace{\{*\}}_{1} + A = 1 + A$$

$$X + Y = \{0\} \times X \cup \{1\} \times Y$$
$$\{(0,x) \mid x \in X\} \cup \{(1,y) \mid y \in Y$$

$$f : A \longrightarrow B$$

$$Tf : TA \longrightarrow TB$$
$$(0,*) \longmapsto (0,*)$$
$$(1,x) \longmapsto (1, fx)$$

$$\eta_A : A \longrightarrow TA$$
$$x \longmapsto (1,x)$$

$$\begin{tikzcd}
x \arrow[rr, mapsto, bend left] \arrow[ddd, mapsto, bend right] & & (1,x) \arrow[ddd, mapsto, bend left] \\
& A \arrow[r, "\eta_A"] \arrow[d, "f"'] & \{*\}+A \arrow[d, "Tf"] \\
& B \arrow[r, "\eta_B"'] & \{*\}+B \\
fx \arrow[rr, mapsto] & & (1, fx)
\end{tikzcd} \quad =$$

$$\mu_A : T(TA) \longrightarrow TA$$
$$(0,*) \longmapsto (0,*)$$
$$(1,(0,*)) \longmapsto (0,*)$$
$$(1,(1,x)) \longmapsto (1,x)$$

Vektorski prostori

$$TX = \left\{ \sum_{i=1}^{n} \lambda_i x_i \;\middle|\; \lambda_i \in \mathbb{R},\ x_i \in X \text{ za } i = 1, \dots, n,\ n \geq 0 \right\}$$

$$= \left\{ \lambda : X \to \mathbb{R} \;\middle|\; \left| \{ x \in X \mid \lambda(x) \neq 0 \} \right| < \infty \right\}$$

$$X \xrightarrow{f} Y \qquad TX \xrightarrow{Tf} TY$$

$$\sum_{i=1}^{n} \lambda_i x_i \longmapsto \sum_{i=1}^{n} \lambda_i f(x_i)$$

$$\begin{tikzcd} \sum \lambda_i x_i \arrow[r, mapsto] & 1 \cdot \left(\sum \lambda_i x_i \right) \arrow[d] \\ & \sum_i \lambda_i \cdot x_i \end{tikzcd}$$

$$\begin{tikzcd} TX \arrow[r, "\eta_{TX}"] \arrow[dr, "id"'] & T^2 X \arrow[d, "\mu_X"] \\ & TX \end{tikzcd}$$

$$\eta_X : X \longrightarrow TX$$

$$x \longmapsto 1 \cdot x$$

$$\mu_X : T(TX) \longrightarrow TX$$

$$\sum_{i=1}^{n} \lambda_i \cdot \left(\sum_{j=1}^{m_i} \lambda'_{ij} \cdot x_{ij} \right) \longmapsto \sum_{i=1}^{n} \sum_{j=1}^{m_i} (\lambda_i \cdot \lambda'_{ij}) \cdot x_{ij}$$

Metrična napolnitev

$T(M,d)$ metrična napolnitev M

$\Vert$

$\{(x_n)_n \mid (x_n)_n$ je Cauchyevo zaporedje v $M\}/\sim$

$f : M \longrightarrow L$ enakomerno zvezna

$Tf : TM \longrightarrow TL$

η

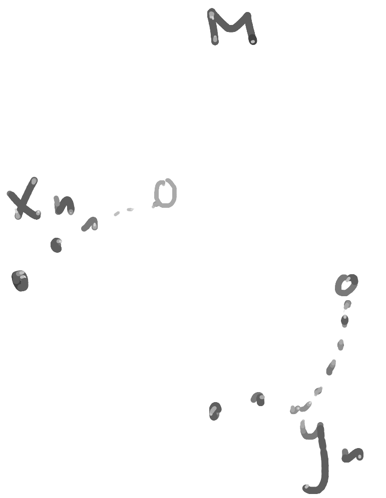
$$\begin{aligned} \eta_M : M &\longrightarrow TM \\ x &\longmapsto [\lambda n.\, x]_\sim \end{aligned}$$

$\mu_M : T(TM) \longrightarrow TM$ izometrija in bijekcija

$[([(x_{m,n})_n])_m]_\sim \qquad [(x_m)_m]$

Metrika na napolnitvi

$$d_{TM}\left([(x_m)_m],\ [(y_n)_n]\right) = \lim_{k\to\infty} d_M(x_k, y_k)$$



Stanje

Čista funkcija $A \longrightarrow B$

prostor stanj S (npr. $S = \mathbb{Z}$ imamo eno spremenljivko tipa Integer)

funkcija, ki uporablja stanje:

$$\begin{tikzcd} S \times A \arrow[r] & S \times B \end{tikzcd}$$

↑ začetno stanje

↑ končno stanje

Curry: $X \times Y \longrightarrow Z \quad \cong \quad Y \longrightarrow (Z^X)$

$Y \to (X \to Z)$

$\hat{\tilde{g}} = g \qquad g(x,y)$

$\tilde{g}(y) = \lambda x . g(x,y)$

Stanje

$$TA = (S \times A)^S$$

$$f : S \times A \longrightarrow S \times B$$

$$A \longrightarrow \underbrace{(S \times B)^S}_{TB}$$

$$f : A \to B$$

$$Tf : (S \times A)^S \longrightarrow (S \times B)^S$$

$$p \longmapsto \lambda s . (\text{let } (s', x) = p s \text{ in } (s', f x))$$

$$\eta_A : A \longrightarrow (S \times A)^S$$

$$x \longmapsto \lambda s . (s, x)$$

$$\mu_A : (S \times (S \times A)^S)^S \longrightarrow (S \times A)^S$$

$$p \longmapsto \lambda s . (\text{let } (s', q) = p s \text{ in } q s')$$

Monade v Haskellu

$$T : \mathrm{Set} \to \mathrm{Set}$$

$$\mathrm{return}_A : A \to TA$$

$$\gg= \; : TA \times (A \to TB) \to TB$$

Enačbe:

$$\mathrm{return}\ x \gg= f \quad = \quad f\,x$$

$$m \gg= \mathrm{return} \quad = \quad m$$

$$m \gg= (\lambda x .\ k\,x \gg= h) \quad = \quad (m \gg= k) \gg= h$$

Monada:

$$Tx = TX$$

$$f : A \to B \qquad Tf : TA \to TB\ ?$$

$$TA \qquad \begin{tikzcd} & B \arrow[dr, "\mathrm{return}_B"] & \\ A \arrow[ur, "f"] \arrow[rr] & & TB \end{tikzcd}$$

$$Tf := \lambda m .\ m \gg= (\mathrm{return}_B \circ f)$$

$$(T\,\mathrm{id}_A)\,m = m \gg= (\mathrm{return}_B \circ \mathrm{id}_A) = m \gg= \mathrm{return}_B = m$$

Monade v Haskellu

$$\eta_A = \mathrm{return}_A$$

$$\begin{tikzcd}
x \arrow[rrr, mapsto] \arrow[ddd, bend right] & & & \mathrm{return}\ x \arrow[dd, mapsto] \\
& A \arrow[r, "\eta_A"] \arrow[d, "f"'] & TA \arrow[d, "Tf"] & \\
& B \arrow[r, "\eta_B"'] & TB & Tf\ (\mathrm{retur}\ x) \\
fx \arrow[rr, mapsto] & & \mathrm{return}\ (f\ x) &
\end{tikzcd}$$

$$\begin{aligned}
Tf\ (\mathrm{return}\ x) &= \mathrm{return}\ x \gg\!= (\mathrm{return} \circ f) \\
&= (\mathrm{return} \circ f)\ x \\
&= \mathrm{return}\ (f\ x)
\end{aligned}$$

Monade

$$\mu_A : T(TA) \longrightarrow TA$$

$$m \gg= id_{TA}$$

TX $\quad x \xrightarrow{f} TY$

m

$m \gg= f$

$T(TA) \quad TA \to TA$

m

do notacija

$$\underline{\underline{m}} \gg= (\lambda x.\ f x \gg= \lambda y. \ \cdots)$$

$$\begin{aligned} \text{do}\ \ & x \leftarrow m \\ & y \leftarrow f x \\ & \cdots \end{aligned}$$

Monade

T, η, μ monada

Haskell monada? $T = T$

return $= \eta$

$$\gg= \; : \; \underset{m}{TA} \times (\underset{f}{A \rightarrow TB}) \longrightarrow TB$$

$$\begin{tikzcd} \underset{m}{TA} \arrow[r, "Tf"] & T(TB) \arrow[r, "\mu_B"] & TB \end{tikzcd}$$

$$m \gg= f \; = \; \mu_B(Tf\, m)$$

Enačba

$$m \gg= (\lambda x.\, k x \gg= \underset{\underset{\lambda y.\, h y}{\parallel}}{h}) \;=\; (m \gg= \underset{\underset{\lambda x.\, k x}{\parallel}}{k}) \gg= \underset{\lambda y.\, h y}{h}$$

$$\begin{array}{ll} \mathrm{do} & x \leftarrow m \\ & y \leftarrow k x \\ & h y \end{array} \qquad\qquad \begin{array}{ll} \mathrm{do} & y \leftarrow (\mathrm{do}\; x \leftarrow m \\ & \qquad\qquad\quad k x\,) \\ & h y \end{array}$$

$$m \gg= f \;=\; \mu(T f\, m)$$

$$m \gg= (\lambda x.\, k x \gg= h) \;=$$

$$\mu(T(\lambda x.\, k x \gg= h)\, m) \;=$$

$$\mu_C(\underbrace{\underbrace{T(\underbrace{\lambda x^{:A}.\, \mu_C(\underbrace{T h (k x)}_{T^2 C})}_{A \longrightarrow T C})}_{T A \longrightarrow T^2 C}\, m}_{T^2 C})$$

$$\overset{?}{=}$$

$$(m \gg= k) \gg= h \;=$$

$$\mu(T h\, (m \gg= k)) \;=$$

$$\mu_C(T \underbrace{h\, (\mu_B(T k\, m))}_{})$$

$$\begin{array}{c} TA \\ m \end{array} \qquad A \xrightarrow{k} TB \qquad B \xrightarrow{h} TC$$

$$TB \xrightarrow{Th} T^2 C$$