Deli & vladaj

Problem P :

→ če je P dovolj majhen problem, ga rešimo neposredno

→ sicer problem razdelimo na podprobleme

$P_1 \quad P_2 \quad \dots \quad P_n$ — manjši od P — istega tipa } deli

→ z enakim postopkom (rekurzivno) rešimo probleme in dobimo rešitve

$R_1 \quad R_2 \quad \dots \quad R_n$

→ rešitve sestavimo v rešitev } vladaj

R

problema P.

Primeri :

Bisekcija : problem : urejena tabela a
v podtabeli a[i:j] poišči indeks elementa x

• podproblem : v pod manjši podtabeli poišči indeks x

Urejanje z zlivanjem :

P : uredi tabelo a dolžine n

podproblemi: P_1: uredi $a[:\frac{n}{2}]$ $\rightsquigarrow$ b urejena

P_2: uredi $a[\frac{n}{2}:]$ $\rightsquigarrow$ c urejena

vladaj: zlij urejeni tabeli b in c

Časovna zahtevnost deli & vladaj v primeru:

- vedno imamo k podproblemov velikosti $\alpha \cdot n$ (n = velikost problema) $0 \leq \alpha < 1$
- deli & vladaj skupaj opravita n^r dela za neki $r > 0$.

Dobimo: $T(n)$ = časovna zahtevnost za problem velikosti n

$$T(1) = 1$$
$$T(n) = n^r + k \cdot T(\alpha n)$$

$$\begin{aligned}
T(n) &= n^r + k \cdot \left((\alpha n)^r + k \cdot T(\alpha^2 n)\right) = \\
&= n^r \cdot (1 + k \cdot \alpha^r) + k^2 T(\alpha^2 n) = \\
&= n^r \cdot (1 + k \cdot \alpha^r) + k^2\left((\alpha^2 n)^r + k \cdot T(\alpha^3 n)\right) = \\
&= n^r (1 + k \cdot \alpha^r + k^2 \cdot \alpha^{2r}) + k^3 T(\alpha^3 n) = \\
&\vdots \\
&= n^r \left(1 + k \cdot \alpha^r + (k \cdot \alpha^r)^2 + \dots + (k \cdot \alpha^r)^j\right) + k^{j+1} T(\alpha^{j+1} n)
\end{aligned}$$

ustavi se, ko $\alpha^{j+1} n = 1$

$$1 + j_0 = \log_\alpha \frac{1}{n} = -\log_\alpha n$$
$$j_0 = -\log_\alpha n - 1 = \log_\alpha \frac{1}{n} - 1$$

$$1 + x + x^2 + \dots + x^j = \frac{1 - x^{j+1}}{1 - x}$$

- $1 + k\alpha^r + \cdots + (k\cdot\alpha^r)^{j_0} = \dfrac{(k\cdot\alpha^r)^{j_0+1} - 1}{(k\alpha^r - 1)}$

- $k^{j_0+1} T(1) = k^{\log_\alpha \frac{1}{n}} = \left(\alpha^{\log_\alpha k}\right)^{\log_\alpha \frac{1}{n}} =$

$$= \left(\alpha^{\log_\alpha \frac{1}{n}}\right)^{\log_\alpha k} = \left(\frac{1}{n}\right)^{\log_\alpha k}$$

$$\Rightarrow O\left(n^r \cdot k^{\log_\alpha \frac{1}{n}} \cdot \alpha^{r\cdot \log_\alpha \frac{1}{n}} + k^{\log_\alpha \frac{1}{n}}\right)$$

$$\Rightarrow O\left(n^r \cdot \left(\frac{1}{n}\right)^r \cdot k^{\log_\alpha \frac{1}{n}} + k^{\log_\alpha \frac{1}{n}}\right)$$

$$\Rightarrow O\left(k^{\log_\alpha \frac{1}{n}}\right)$$

Preverimo: urejanje z zlivanjem: $k=2$ $\alpha=\frac{1}{2}$ $r=1$

Še enkrat geometrijska:

$$n^r\cdot\left(1+\cdots+(k\alpha^r)^{j_0}\right) = n^r\cdot\frac{1-(k\alpha^r)^{j_0+1}}{1-k\alpha^r} =$$

1) $k\alpha^r < 1 \Rightarrow n^r \underbrace{\left(1+\cdots+(k\alpha^r)^{j_0}\right)}_{< \frac{1}{1-(k\alpha^r)}} \in O(n^r)$

2) $k\alpha^r > 1 \Rightarrow \quad n^r \cdot \dfrac{(k\alpha^r)^{j_0+1}-1}{k\alpha^r - 1} = n^r \cdot (k\alpha^r)^{\cancel{1+}\log_\alpha \frac{1}{n} \cancel{-1}}$

$$= n^r \cdot (k\alpha^r)^{\log_\alpha \frac{1}{n}} =$$

$$\cancel{n^r} \cdot k^{\log_\alpha \frac{1}{n}} \cdot \cancel{\left(\frac{1}{n}\right)^r} = k^{\log_\alpha \frac{1}{n}}$$

3) $k\alpha^r = 1 \Rightarrow$

$$n^r(\underbrace{1+1+\dots+1}_{j_0}) = n^r \cdot j_0$$

<u>Odgovor:</u>

1) $k\alpha^r < 1$: $\quad \Theta(n^r + n^{-\log_\alpha k}) = \underline{\underline{\Theta(n^r)}}$

$\log_\alpha k + r > 0$

$r > -\log_\alpha k$

2) $k\alpha^r = 1$: $\quad \Theta\left(n^r \cdot \left(\log_\alpha \frac{1}{n} - 1\right) + n^{-\log_\alpha k}\right)$

$\log_\alpha k + r = 0$

$r = -\log_\alpha k$

$$\Theta(n^r \cdot (-\log_\alpha n - 1) \cdot n^r)$$

$$\Theta(n^r \cdot (-\log_\alpha n)) = \Theta(n^r \cdot \log_{1/\alpha} n)$$

$$= \mathcal{O}(n^r \cdot \log n)$$

3) $k\alpha^r > 1$: $\mathcal{O}\left(k^{\log_\alpha \frac{1}{n}} + \left(\frac{1}{n}\right)^{\log_\alpha k}\right)$

$$\mathcal{O}\left(\left(\frac{1}{n}\right)^{\log_\alpha k} + n^{-\log_\alpha k}\right)$$

$$\mathcal{O}\left(n^{-\log_\alpha k}\right) = \mathcal{O}\left(n^{\log_{1/\alpha} k}\right)$$

Končni rezultat:

Problem velikosti n razdelimo na k podproblemov, vsak od njih je velik n/b. Deli in vladaj opravita skupaj n^r korakov.

Časovna zahtevnost:

$$T(1) = 1 \qquad T(n) = n^r + k \cdot T\left(\frac{n}{b}\right)$$

1) Če je $k < b^r$: $T(n) \in \mathcal{O}(n^r)$

2) Če je $k = b^r$: $T(n) \in \mathcal{O}(n^r \cdot \log_b n)$

3) Če je $k > b^r$: $T(n) \in \mathcal{O}(n^{\log_b k})$

Strassenovo množenje matrik

Množimo matrike velikosti $n \times n$

- Običajno množenje: $A \cdot B = C$

$$\underbrace{C_{i,j}}_{n^2 \text{ členov}} = \underbrace{\sum_{k=1}^{n} A_{i,k} \cdot B_{k,j}}_{O(n) \text{ operacij}} \Rightarrow O(n^3)$$

- Bločno množenje:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$\underbrace{C_{ij}}_{4 \text{ bloki}} = \underbrace{A_{i1} \cdot B_{1j} + A_{i2} \cdot B_{2j}}_{O(n^2) \text{ seštevanj}}$$

Deli in vladaj:

$k = 8$, $b = 2$, $r = 2$

$k > b^r$

$$T(n) \in O(n^{\log_b k}) = O(n^3)$$

- Strassen:

The Strassen algorithm defines instead new matrices:

$$
\begin{aligned}
\mathbf{M}_1 &:= (\mathbf{A}_{1,1} + \mathbf{A}_{2,2})(\mathbf{B}_{1,1} + \mathbf{B}_{2,2}) \\
\mathbf{M}_2 &:= (\mathbf{A}_{2,1} + \mathbf{A}_{2,2})\mathbf{B}_{1,1} \\
\mathbf{M}_3 &:= \mathbf{A}_{1,1}(\mathbf{B}_{1,2} - \mathbf{B}_{2,2}) \\
\mathbf{M}_4 &:= \mathbf{A}_{2,2}(\mathbf{B}_{2,1} - \mathbf{B}_{1,1}) \\
\mathbf{M}_5 &:= (\mathbf{A}_{1,1} + \mathbf{A}_{1,2})\mathbf{B}_{2,2} \\
\mathbf{M}_6 &:= (\mathbf{A}_{2,1} - \mathbf{A}_{1,1})(\mathbf{B}_{1,1} + \mathbf{B}_{1,2}) \\
\mathbf{M}_7 &:= (\mathbf{A}_{1,2} - \mathbf{A}_{2,2})(\mathbf{B}_{2,1} + \mathbf{B}_{2,2})
\end{aligned}
$$

only using 7 multiplications (one for each M_k) instead of 8. We may now express the $C_{i,j}$ in terms of M_k:

$$
\begin{aligned}
\mathbf{C}_{1,1} &= \mathbf{M}_1 + \mathbf{M}_4 - \mathbf{M}_5 + \mathbf{M}_7 \\
\mathbf{C}_{1,2} &= \mathbf{M}_3 + \mathbf{M}_5 \\
\mathbf{C}_{2,1} &= \mathbf{M}_2 + \mathbf{M}_4 \\
\mathbf{C}_{2,2} &= \mathbf{M}_1 - \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_6
\end{aligned}
$$

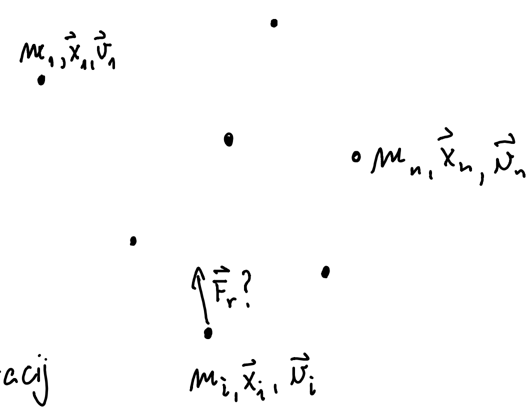
$k = 7 \qquad b = 2 \qquad r = 2$

$7 > 2^2$

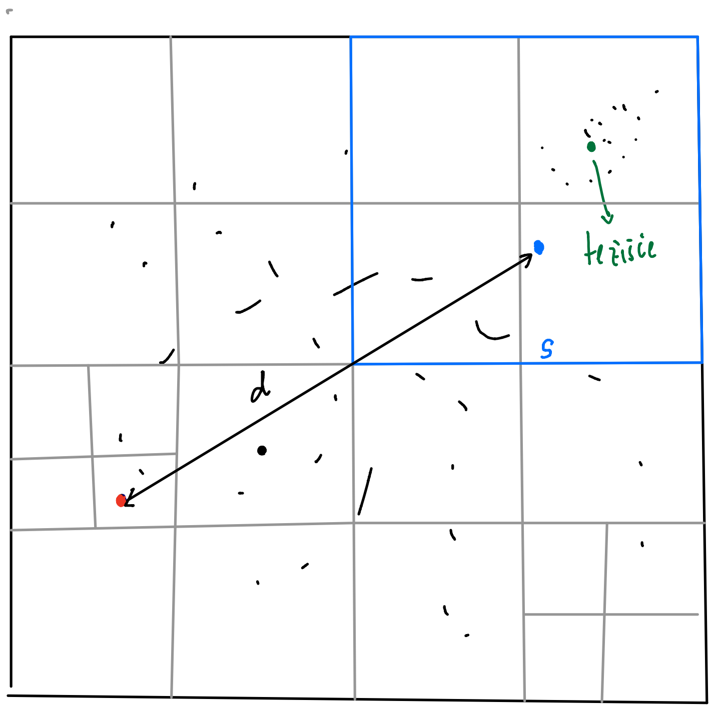
$$T(n) = O(n^{\log_b k}) = $$
$$= O(n^{\log_2 7}) \subseteq O(n^{2.81})$$

Barnes-Hut

Simuliramo sile med n točkami po gravitacijskem zakonu.



Za vsak par teles i, j izračunamo silo $\vec{F}_{ij}$ $\Rightarrow$ $O(n^2)$ operacij



če $s<\theta\cdot d$, uporabimo težišče, sicer delimo

$O(n \log n)$!