Topology in the topos of countable reals

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(b) and topology is very strange,

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… just as there is an alternative geometry

(a) in which the parallel postulate fails
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Talk overview

- Uncountability of $\mathbb{R}$
- The topos of countable reals
- Topological phenomena in the topos

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Theorem 3. *A sequence-avoiding set is uncountable.*

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The above proof is intuitionistic.

Note: *Intuitionistic mathematics* is mathematics without excluded middle and the axiom of choice. Do not confuse this with "mathematics which denies excluded middle and the axiom of choice". (Likewise, a non-abelian group isn't one satisfying $\forall x, y \,.\, xy \neq yx$.)

Cantor's diagonal argument

Theorem 4 (Cantor 1879). $\{0,1\}^{\mathbb{N}}$ *is sequence-avoiding.*

Proof. The sequence $a : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$ is avoided by $x := (n \mapsto 1 - a_n(n))$. $\square$

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Theorem 5 (Cantor 1874). $\mathbb{R}$ *is sequence-avoiding.*

Proof. Given $a : \mathbb{N} \to \mathbb{R}$ define $[u_0, v_0] := [0,1]$ and

$$[u_{n+1}, v_{n+1}] := \begin{cases} [u_n, (4u_n + v_n)/5] & \text{if } a_n > (3u_n + 2v_n)/5, \\ [(u_n + 4v_n)/5, v_n] & \text{if } a_n < (2u_n + 3v_n)/5. \end{cases}$$

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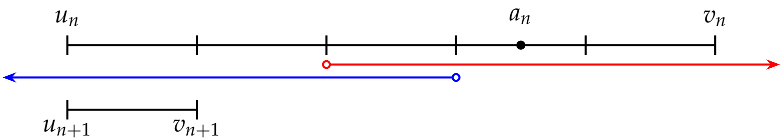

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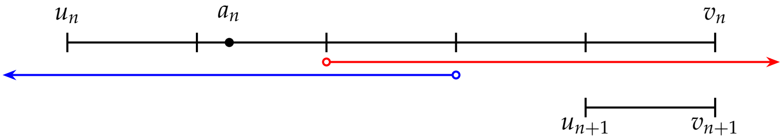

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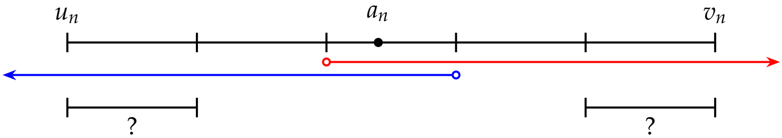

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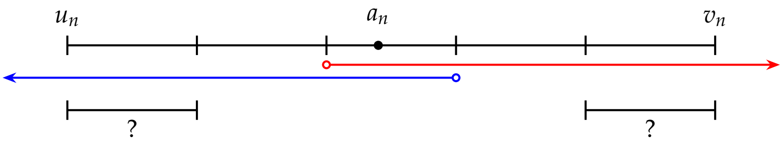

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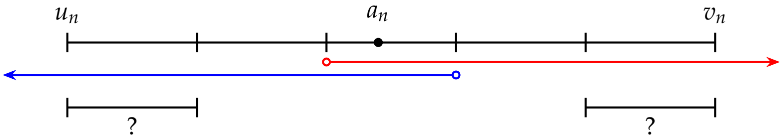

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Question. *Can $\mathbb{R}$ be proved uncountable without these principles?*

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Theorem 6. *There is a topos in which $\mathbb{R}$ is countable.*

Proof. See A. Bauer & J.E. Hanson: "The topos of countable reals". $\square$

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> *"An elementary topos is a category which has properties making it resemble the category of sets. Elementary toposes can be used as models of intuitionistic higher-order logic."* (Wikipedia)

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Skolem's "paradox": *external* and *internal* countability may disagree.

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Theorem 7. *Every realizability topos validates countable choice.*

Recall: Cantor's proof works with countable choice.

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Theorem 9 (Miller). *There is a sequence $\boldsymbol{\mu} : \mathbb{N} \to [0,1]$ such that, for all $n \in \mathbb{N}$, if the n-th Turing machine $\boldsymbol{\mu}$-computes $x \in [0,1]$ then $\boldsymbol{\mu}(n) = x$.*

Proof. See J. S. Miller "Degrees of Unsolvability of Continuous Functions". The proof uses Kakutani's fixed-point theorem for upper semicontinuous multivalued maps on $[0,1]^{\mathbb{N}}$. $\square$

The parameterized realizability topos $\mathcal{T}$

- We defined a new kind of topos called a *parameterized realizability topos*.
- We constructed a specific such topos $\mathcal{T}$ that captures Miller's computability.

Theorem 10. *In $\mathcal{T}$ there is a surjection $\mathbb{N} \to \mathbb{R}$.*

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Which $\mathbb{R}$?

There are three ways to complete the ordered field $\mathbb{Q}$:

- Cauchy reals $\mathbb{R}_C$: by limits of Cauchy sequences.
- Dedekind reals $\mathbb{R}$: by (two-sided) Dedekind cuts.
- MacNeille reals $\mathbb{R}_M$: by suprema of inhabited bounded sets.

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Intuitionistically, we prefer $\mathbb{R}$:

- $\mathbb{R}_C \subseteq \mathbb{R} \subseteq \mathbb{R}_M$.
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Theorem 11. *In $\mathcal{T}$, $\mathbb{R}_C$ is uncountable, $\mathbb{R}$ is countable, and $\mathbb{R}_M$ is uncountable.*

What else is countable in $\mathcal{T}$?

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Theorem 12. *In $\mathcal{T}$, Hilbert's cube $[0,1]^{\mathbb{N}}$ is countable.*

Brouwer's fixed-point theorem in $\mathcal{T}$

Theorem 13 (intuitionistic). *If Hilbert's cube $[0,1]^{\mathbb{N}}$ is countable then every $f : [0,1]^n \to [0,1]^n$ has a fixed point.*

Proof. Note that $([0,1]^n)^{\mathbb{N}} \cong [0,1]^{n\times\mathbb{N}} \cong [0,1]^{\mathbb{N}}$ is countable, so let $e : \mathbb{N} \to ([0,1]^n)^{\mathbb{N}}$ be a surjection. Define $g : \mathbb{N} \to [0,1]^n$ by $g(k) := f(e(k)(k))$. There is $m \in \mathbb{N}$ such that $e(m) = g$, and then $e(m)(m) = g(m) = f(e(m)(m))$ is a fixed point of f. □

Corollary 14. *In $\mathcal{T}$, Brouwer's fixed-point theorem holds for all maps.*

Are all maps continuous?

Definition 15. A map $f : \mathbb{R} \to \mathbb{R}$ **jumps at** 0 if there is $\epsilon > 0$ such that $f(y) - f(x) > \epsilon$ for all $x < 0 < y$.

Theorem 16 (intuitionistic). *If there is a map with a jump then $\mathbb{R}$ is uncountable.*

Corollary 17. *In $\mathcal{T}$ maps $\mathbb{R} \to \mathbb{R}$ do not jump.*

Question. *Are all maps $\mathbb{R} \to \mathbb{R}$ continuous in $\mathcal{T}$?*

Singular covers and failure of compactness

Let $0 < \epsilon < 1$.

Definition 18. An ϵ**-singular cover** is a sequence of open intervals $(a_0, b_0), (a_1, b_1), \ldots$ which cover $[0, 1]$ and satisfy $\sum_{k \geq 0}(b_k - a_k) \leq \epsilon$.

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Theorem 19 (intuitionistic). *If $[0, 1]$ is countable then it has an ϵ-singular cover.*

Proof. Given a surjection $e : \mathbb{N} \to [0, 1]$, define $a_k := e(k) - \epsilon \cdot 2^{-k-2}$ and $b_k := e(k) + \epsilon \cdot 2^{-k-2}$. $\square$

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Theorem 20. *In $\mathcal{T}$:*

(a) *$[0, 1]$ is not Heine-Borel compact,*

(b) *$[0, 1]$ is Cauchy-complete and totally bounded.*

Proof. (a) The ϵ-singular cover has no finite subcover. (b) Standard. □

Compactness revisited

Theorem 21. *In $\mathcal{T}$, every cover of $[0,1]$ by intervals $(a_0, b_0), (a_1, b_1), (a_2, b_2), \ldots$* ***with rational endpoints*** *has a finite subcover.*

Compactness revisited

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The proof relies on a construction of [Orevkov 1963]:

Lemma 22 (intuitionistic). *Suppose (a_0, b_0), (a_1, b_1), (a_2, b_2), $\ldots$ are intervals with rational endpoints. There is a continuous map*

$$h : \left([0,1] \cap \bigcup_{k \geq 0} (a_k, b_k)\right)^2 \to [0,1]^2$$

such that: h has a fixed point if, and only if, $\bigcup_{k \leq n} (a_k, b_k)$ covers $[0,1]$ for some $n \in \mathbb{N}$.

Proof. Rationality of endpoints is used to decide how the intervals overlap. □

Five stages of accepting this talk

1. *Denial:* The speaker is a logician.
2. *Anger:* Stop ruining topology!
3. *Bargaining:* It's just a model.
4. *Depression:* What if I am trapped in a model?
5. *Acceptance:* There are many worlds of mathematics.