Turing Degrees in Synthetic Computability

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Heyting Day 2025 – Models of intuitionism and computability
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Talk outline

1. What is synthetic mathematics?
2. Synthetic computability
3. Turing degrees in synthetic computability

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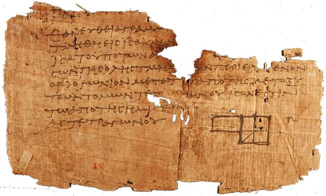

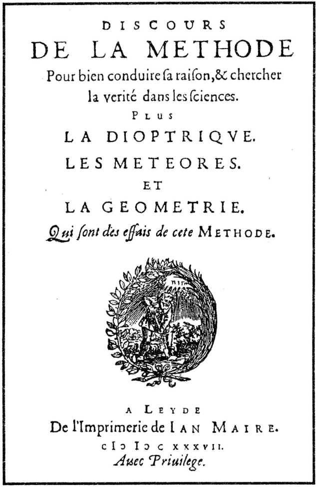

DISCOURS
DE LA METHODE
Pour bien conduire ſa raiſon, & chercher la verité dans les ſciences.
PLUS
LA DIOPTRIQVE.
LES METEORES.
ET
LA GEOMETRIE.
Qui ſont des eſſais de cete METHODE.

A LEYDE
De l'Imprimerie de IAN MAIRE.
cIↄ Iↄ c xxxvii.
Auec Priuilege.

Synthetic:

- Basic objects are taken as *primitive*.
- Their properties & relations are *axiomatized*.
- We work within the given axiomatic system.

Analytic:

- Basic objects are *constructed* from other objects.
- Their properties & relations are *deduced*.
- We work in a foundational system.

Branches of synthetic mathematics

- Non-standard analysis
- Synthetic differential geometry
- Synthetic domain theory
- Synthetic topology
- Synthetic computability
- Synthetic homotopy theory
- Synthetic algebraic geometry

Synthetic		Analytic
Theory	interpretation →	Model
Intuitionistic Higher-order logic (IHOL)	internal language →	Topos
IHOL & specific axioms	→	Specific topos

A paradise for everyone

- Synthetic differential geometry:
  "All maps are smooth."
- Synthetic topology:
  "All maps are continuous."
- Effective topos:
  "All maps are computable."
  "All maps are continuous."
  "There is an unbounded map $[0, 1] \to \mathbb{R}$*."*
  "There is a set which is neither finite nor infinite."
  "Trichotomous ordinals form a set."
- Infinite-time effective topos: "$\mathbb{N}^{\mathbb{N}}$ *embeds into* $\mathbb{N}$*."*
- Parametric realizability: "$\mathbb{R}$ *is countable."*

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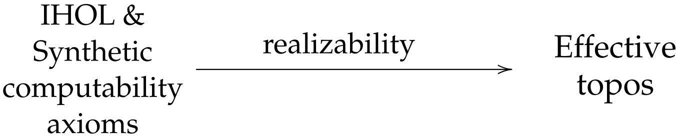
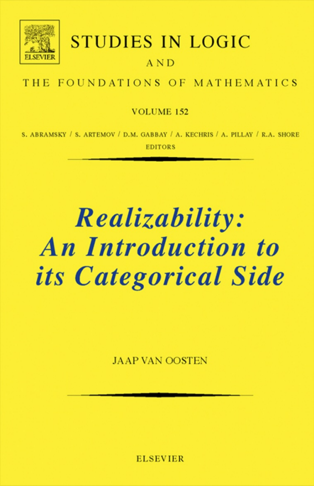
Synthetic computability



The effective topos



The axioms of synthetic computability

- Intuitionistic higher-order logic:
  - an object of truth values Ω
  - natural numbers $\mathbb{N}$
  - constructions: products, sums, exponents, quotients, …
  - sufficient for "everyday math"
- Dependent choice
  - For most applications countable choice suffices.
  - In few cases we use choice on $\neg\neg$-stable subsets of $\mathbb{N}$.
- Markov principle:

  If a binary sequence is not all 0*'s, it contains a* 1*.*

- Enumerability axiom:

  There are countably many countable subsets of $\mathbb{N}$*.*

The synthetic method

1. Take a classic theorem in computability theory.
2. Rephrase it as a fact about the effective topos.
3. Find an internal statement whose interpretation is the fact.
4. Abstract and generalize the statement to expose its essence.
5. Give a synthetic proof.

Bonus:

- Prove the synthetic statement in pure IHOL.
- Exhibit non-trivial instances using the synthetic axioms.

Rice's theorem

Recall:

- Let φ_n be the n-th partial computable map $\mathbb{N} \rightharpoonup \mathbb{N}$.
- $S \subseteq \mathbb{N}$ is an ***index set*** when $m \in S \wedge \varphi_m = \varphi_n \Rightarrow n \in S$.

Proposition (Rice's theorem)

The only computably decidable index sets are $\emptyset$ and $\mathbb{N}$.

Synthetic Rice's theorem

Definition

A set A has the ***fixed-point property*** if every $f : A \to A$ has a fixed point.

Proposition (Synthetic Rice's theorem)

If A has the fixed point property then every map $A \to \mathsf{2}$ is constant.

Proof.

Given $f : A \to \mathsf{2}$ and any $x, y \in A$ we show that $f(x) = f(y)$. Define $g : A \to A$ by $g(z) := \mathsf{if}\ f(z) = f(y)\ \mathsf{then}\ x\ \mathsf{else}\ y$. There is $u \in A$ such that $u = g(u)$. Either $f(u) = f(y)$ or $f(u) \neq f(y)$. If $f(u) = f(y)$ then $u = g(u) = x$ and $f(x) = f(u) = f(y)$. If $f(u) \neq f(y)$ then $u = g(u) = y$ and so $f(u) = f(y)$, a contradiction, hence again $f(x) = f(y)$. □

Bonus: the proof is pure; and in synthetic computability, ω-algebraic pointed ω-cpos have the fixed-point property.

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What is an oracle?

Definition

A ***partial oracle*** is a pair (A_0, A_1) of disjoint subsets of $\mathbb{N}$.

Think of A_0 and A_1 as the negative and positive answers.
The object of partial oracles

$$\mathbb{O} := \{(A_0, A_1) \in \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}) \mid A_0 \cap A_1 = \emptyset\}$$

ordered by component-wise inclusion is

- *directed-complete (dcpo)*: directed subsets have supremuma,
- *ω-algebraic*: pairs of disjoint finite sets form a base.

An oracle is total when $A_0 = \mathbb{N} \setminus A_1$ and $A_1 = \mathbb{N} \setminus A_0$.
(Caveat: $A_0 \cup A_1 = \mathbb{N}$ yields a decidable oracle!)

What is a Turing reduction?

Recall:

- Classically, a map $r : \mathbb{O} \to \mathbb{O}$ is a ***Turing reduction*** if it is realized by a Turing machine.
- A map $f : D \to E$ between ω-algebraic dcpos is ***Scott continuous*** if it preserves directed suprema. Its ***graph*** is

$$\Gamma_f := \{(d, e) \in \mathcal{K}(D) \times \mathcal{K}(E) \mid e \leq f(d)\}$$

where $\mathcal{K}(D)$ are the compact elements in D.

Definition

A ***(Turing) reduction*** is a Scott continuous map *whose graph is countable.*

Incomparable oracles

Proposition (Classical Kleene-Post Theorem)

There exist incomparable oracles.

- Starting from empty oracles, the proof builds oracles $(x, y) \in \mathbb{O} \times \mathbb{O}$ by finite extension method, using the n-th extension to ensure that the n-th Turing machine does not reduce x to y or vice versa.
- The proof relies on decidability of the halting set.

Proposition (Baire category theorem)

In an ω-algebraic dcpo, a countable intersection of sets with dense interiors is dense.

(The proof uses dependent choice.)

Synthetic Kleene-Post I

Proposition (Synthetic)

There is an enumeration $\rho_0, \rho_1, \rho_2, \ldots$ of Turing reductions $\mathbb{O} \to \mathbb{O}$.

(The proof uses the Enumerability axiom.)

Do the sets

$$R_n := \{(x, y) \in \mathbb{O} \times \mathbb{O} \mid \rho_n(x) \neq y \wedge \rho_n(y) \neq x\}$$

have dense interiors for the Scott topology on $\mathbb{O}$?
Not quite…

An interlude on modalities

A Lawvere-Tierney ***closure*** operator $j : \Omega \to \Omega$ induces a ***modal operator*** $\Box$ taking each object A to

$$\Box A := \{S \subseteq A \mid j(\exists x \in A \,.\, \forall y \in A \,.\, y \in S j(x = y))\}.$$

There is a least $j : \Omega \to \Omega$ such that

$$\forall f \in \mathsf{2}^{\mathbb{N}} \,.\, j((\exists n \,.\, f(n) = 1) \vee \neg(\exists n \,.\, f(n) = 1)).$$

Henceforth we use this j and the associated $\Box$.
Intuition:

- $A \to B$ are the *computable* maps,
- $A \to \Box B$ are maps *computable relative to the Halting oracle*.

Synthetic Kleene-Post 2

Let D be an ω-algebraic dcpo and $R \subseteq D$ with dense interior. Using countable choice, we obtain a witness for density

$$h : \mathcal{K}(D) \to \mathcal{K}(D)$$

such that $d \leq h(d)$ and $\uparrow h(d) \subseteq R$ for all $d \in \mathcal{K}(D)$.

Given such witnesses $h_0, h_1, h_2, \ldots$ for $R_0, R_1, R_2, \ldots$, the intersection $\bigcap_n R_n$ is inhabited above any $d_0 \in \mathcal{K}(D)$ by $\sup_n d_n$ where $d_{n+1} = h_n(d_n)$.

Alas, in the case of oracle reductions we only have

$$h : \mathcal{K}(D) \to \Box\mathcal{K}(D).$$

Synthetic Kleene-Post 3

Definition

$R \subseteq D$ has $\Box$***-dense interior*** if there is a map $h : \mathcal{K}(D) \to \Box\mathcal{K}(D)$ such that, for every $d \in \mathcal{K}(D)$, $d \leq h(d)$ and $\uparrow h(d) \subseteq R$.

Proposition

In an ω-algebraic dcpo a countable intersection of sets with $\Box$-dense interiors is dense.

Bonus: The theorem is pure, and examples abound.

Synthetic Kleene-Post 4

Corollary (Synthetic Post-Kleene Theorem)

There exist incomparable oracles.

Proof.

Apply the previous proposition:

- The ω-algebraic dcpo is $\mathbb{O} \times \mathbb{O}$.
- The modal operator $\Box$ is the halting modality.
- The sets $R_n = \{(x, y) \in \mathbb{O} \times \mathbb{O} \mid \rho_n(x) \neq y \wedge \rho_n(y) \neq x\}$.
  (It is not hard to verify that they have $\Box$-dense interiors.)

$\square$

Ongoing work: Synthetic Friedberg-Muchnik

Proposition

*There exist incomparable **computably enumerable** oracles.*

The classic proof uses the *finite injury priority method*.

Definition

A subset $R \subseteq D$ has ***fallback-dense interior*** when for every $d \in \mathcal{K}(D)$ there are a ***candidate*** $c \in \mathcal{K}(D)$, and an open ***injury set*** $I \subseteq D$ such that:

- $\uparrow c \setminus I \subseteq R$, and
- for every $d \in I \cap \mathcal{K}(D)$ there is a ***fallback*** $f \in \mathcal{K}(D)$ such that $d \leq f$ and $\uparrow f \subseteq R$.

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Proposition

In an ω-algebraic ω-cpo, a countable intersection of sets with fallback-dense interiors for the Lawson topology is dense.

Synthetic Friedberg-Muchnik: consider the ω-algebraic lattice of countable subsets of $\mathbb{N}$.

"Time will tell how far this development will go."

Jaap van Oosten