Classically laughable theorems

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The Topos Institute Colloquium
April 2, 2026

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- In classical mathematics it has only trivial instances.
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- In classical mathematics it has only trivial instances.
- Receives content in a variety of synthetic mathematics.
- Classically irritating mathematics:
  - Classically trivial concepts.
  - Classically false, intuitionistically undecided statements.
  - Axioms that defy common mathematical sense.

Why should classical mathematicians be interested?

- Explore other worlds of mathematics.
- A jolt to your mathematical intuition.
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In this talk:

- Examples of laughable theorems and concepts.
- They receive content in synthetic computability.
- A slogan for you to take home.

First example: Lawvere's fixed-point theorem

Theorem (Lawvere)

If $e : A \to B^A$ is surjective then every $f : B \to B$ has a fixed point.

Proof. There is $x \in A$ such that $e(x) = \lambda y. f(e(y)(y))$. Then $e(x)(x) = f(e(x)(x))$.

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The contrapositive form is *not* laughable:

Theorem (Cantor)

There is no surjection $A \to \Omega^A$.

The axioms of synthetic computability

- Intuitionistic higher-order logic:
  - an object of truth values Ω
  - natural numbers $\mathbb{N}$
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The enumerability axiom is fairly irritating, but is valid in the effective topos:

"Computably enumerable subsets of $\mathbb{N}$ *are computably enumerable."*

Use a standard enumeration $W_n = \{k \in \mathbb{N} \mid \varphi_n(k){\downarrow}\}$.

Lawvere's theorem in synthetic computability

Theorem (Intuitionistic)

If an ω-cpo D has a countable base B then so does its power $D^{\mathbb{N}}$.

Proof idea. A base for $D^{\mathbb{N}}$ consists of finitely supported sequences in $B^{\mathbb{N}}$.

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Corollary

If D is an ω-cpo with a countable base then $D^{\mathbb{N}}$ is countable.

A. Bauer, *On fixed-point theorems in synthetic computability*, Tbilisi Math. J. 10(3): 167–181.

A laughable concept

Theorem

A space X is compact if, and only if, the map $\forall_X : \mathcal{O}X \to \mathbb{S}$ is continuous.

- $\mathcal{O}X$ is the lattice of opens with Scott topology,
- $\mathbb{S} = \{\bot, \top\}$ is the Sierpinski space,
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A space X is **overt** when the map $\exists_X : \mathcal{O}X \to \mathbb{S}$ is continuous.

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Theorem (Classical)

Every space is overt.

Overtness computably

Definition

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A computably non-overt space:

- The subspace $\{\alpha\} \subseteq \mathbb{R}$ with $\alpha \in \mathbb{R}$ non-computable.
- If we could semidecide whether $(p, q) \cap \{\alpha\}$ is inhabited for $p, q \in \mathbb{Q}$, we could enumerate the lower and upper rational bounds of α, which would make α computable.

Spreen spaces and the KLST theorem

Definition

A **Spreen space** is a space X in which every point separated from an overt subset by a semidecidable subset is also separated from it by an open subset.

Theorem (Intutionistic)

Every map from an overt Spreen space to a regular space is pointwise continuous.

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Proposition (Classical)

Spreen spaces are the discrete spaces.

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Proposition (Classical)

Spreen spaces are the discrete spaces.

Proposition (Synthetic computability)

Countably based sober spaces are Spreen spaces.

Andrej Bauer: *Spreen spaces and the synthetic Kreisel-Lacombe-Shoenfield-Tseitin theorem*, J. of Log. and Anal. 17:FDS3 (2025) 1–24

A guiding principle for synthetic mathematics

"Prove intuitionistically. Apply synthetically."

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Can we apply the slogan in other varieties of synthetic mathematics?

- synthetic differential geometry
- synthetic algebraic geometry
- univalent mathematics and homotopy type theory