Izrek o negibni točki

D domena, f : D → D zvezna. Tedaj obstaja najmanjša negibna točka f:

1) Obstaja $x \in D$, da $f(x) = x$
2) Če tudi $f(y) = y$, potem $x \leq y$.

Dokaz: Ideja:

$$\underset{x_0}{\perp} \leq \underset{x_1}{f(\perp)} \leq \underset{x_2}{f(f(\perp))} \leq \underset{x_3}{f(f(f(\perp)))} \leq \dots \leq \sup.$$

ker $\perp \leq f(\perp)$ in f monotona

!!!

Definiramo: $x_0 = \perp$
$x_{n+1} = f(x_n)$ za $n \geq 0$

Trdimo, da je $(x_n)_n$ veriga, se pravi $\forall n.\ x_n \leq x_{n+1}$. Indukcija na n:

- $n = 0$: $x_0 = \perp \leq f(\perp) = x_1$
- korak: če $x_n \leq x_{n+1}$, potem $\underset{\substack{\| \\ x_{n+1}}}{f(x_n)} \leq \underset{\substack{\| \\ x_{n+2}}}{f(x_{n+1})}$ ker f monotona

Naj bo $x = \bigvee_{n \in \mathbb{N}} x_n$.

Dokaz

$\mathbb{N} = \{0,1,2,3,4,5,\dots\}$

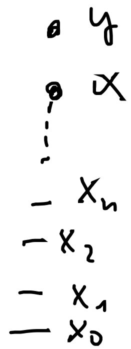
Trdimo $x = f(x)$:

$$f(x) = f\Big(\bigvee_{n\in\mathbb{N}} x_n\Big) \underset{\uparrow \atop f \text{ zvezna}}{=} \bigvee_{n\in\mathbb{N}} f(x_n) = \underbrace{\bigvee_{n\in\mathbb{N}} x_{n+1}}_{\bigvee\{x_1,x_2,x_3,x_4,\dots\}} = \underbrace{\bigvee_{n\in\mathbb{N}} x_n}_{\bigvee\{x_0,x_1,x_2,x_3,\dots\}} = x.$$

$x_0 = \perp$

Denimo $y = f(y)$. Dokažimo $x \le y$:

$$x \le y \iff \bigvee_{n\in\mathbb{N}} x_n \le y \underset{\uparrow \atop \text{po def. supr.}}{\iff} \forall n.\, x_n \le y$$

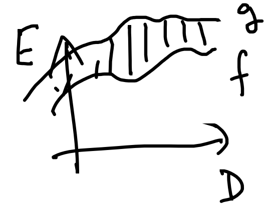


Z indukcijo pokažimo $\forall n.\, x_n \le y$:

- $n = 0$: $x_0 = \perp \le y$ ✓
- korak: Če $x_n \le y$ potem $\underset{f \text{ monotona}}{}$ $\underset{x_{n+1}}{f(x_n)} \le f(y) = y$.

■

Operator fix



$f : D \to D$ zvezna $\Rightarrow$ najmanjša negibna

Definiramo: D, E domeni

$$[D \to E] = \{ f : D \to E \mid f \text{ zvezna} \}$$

$$f \leq_{D \to E} g \iff \forall x \in D.\ f(x) \leq g(x)$$

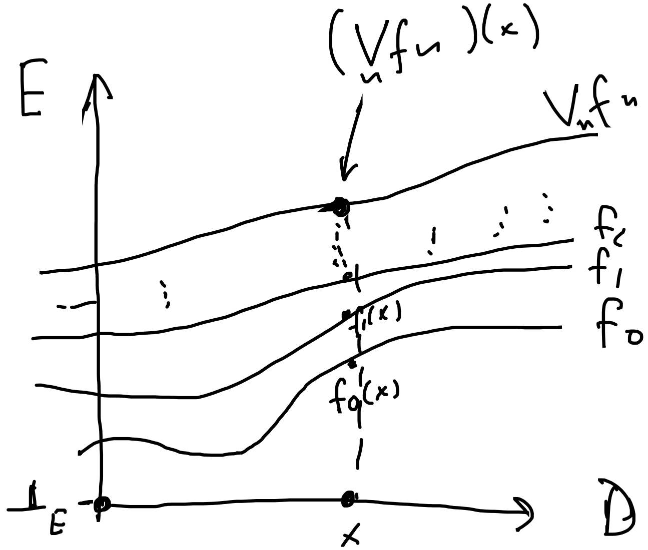
Trdimo: $([D \to E], \leq_{D \to E})$ je domena

(1) $\leq_{D \to E}$ delna ureditev (vaja)

(2) $\bot_{D \to E}$ je funkcija $x \mapsto \bot_E$

(3) $f_0 \leq f_1 \leq f_2 \leq \dots$ veriga v $[D \to E]$

$$\Big(\bigvee_{n \in \mathbb{N}} f_n\Big)(x) = \bigvee_{n \in \mathbb{N}} f_n(x)$$



Operator fix

Definiramo: $\quad fix_D : [D \to D] \to D$

$$f \longmapsto \bigvee_{n \in \mathbb{N}} f^n(\bot)$$

$$f^n(x) = \underbrace{f(f(f \cdots (f}_{n}(x) \cdots)$$

Izrek: fix_D je zvezna funkcija.

Dokaz: (1) fix_D je monotona:

Denimo $f \leq_{D \to D} g$. Tedaj $f^n(\bot) \leq g^n(\bot)$, ker:

- $n = 0 \qquad \bot \leq \bot$
- korak: če $f^n(\bot) \leq g^n(\bot)$ potem $f^{n+1}(\bot) \leq f(g^n(\bot)) \overset{\text{ker } f \leq g}{\leq} g(g^n(\bot)) = g^{n+1}(\bot)$

$$fix_D(f) = \bigvee_{n \in \mathbb{N}} f^n(\bot) \underset{\text{ker } f^n(\bot) \leq g^n(\bot)}{\leq} \bigvee_{n \in \mathbb{N}} g^n(\bot) = fix_D(g).$$

- (2) fix_D ohranja sup verig: $\quad \mathrm{fix}\, g = \bigvee_n g^n \bot$

naj bo $f_0 \leq f_1 \leq f_2 \leq \ldots$ verige v $[D \to D]$:

$$\mathrm{fix}_D\Big(\bigvee_{n\in\mathbb{N}} f_n\Big) = \bigvee_{m\in\mathbb{N}} \Big(\bigvee_{n\in\mathbb{N}} f_n\Big)^m(\bot)$$

Dokažujemo:

$$\bigvee_m \Big(\bigvee_n f_n\Big)^m(\bot)$$

$$\geq$$

$$\bigvee_n \bigvee_m f_n^m(\bot)$$

$$\geq \checkmark \qquad \leq\ ?!$$

$$\Big(\bigvee_{n\in\mathbb{N}} \mathrm{fix}_D\, f_n\Big) = \bigvee_{n\in\mathbb{N}} \bigvee_{m\in\mathbb{N}} f_n^m(\bot)$$

$$\left.\begin{aligned} f_k &\leq \bigvee_n f_n \\ \mathrm{fix}\, f_k &\leq \mathrm{fix}\Big(\bigvee_n f_n\Big) \\ \bigvee_k \mathrm{fix}\, f_k &\leq \mathrm{fix}\Big(\bigvee_n f_n\Big) \end{aligned}\right\}$$

g monotona

$$x_k \leq \bigvee_n x_n$$

$$g(x_k) \leq g\Big(\bigvee_n x_n\Big)$$

$$\bigvee_k g(x_k) \leq g\Big(\bigvee_n x_n\Big)$$

- Dokazujemo: $\bigvee_m \left(\bigvee_n f_n\right)^m(\bot) \leq \bigvee_n \bigvee_m f_n^m(\bot)$

$$\boxed{\begin{array}{c} \bigvee_n x_n \leq y \\ \Updownarrow \\ \forall n.\, x_n \leq y \end{array}}$$

Torej: za vsak m je

$$\left(\bigvee_n f_n\right)^m(\bot) \leq \bigvee_n \bigvee_m f_n^m(\bot)$$

$$\left(\bigvee_i f_i\right)^m(\bot) \leq \bigvee_j \bigvee_k f_j^k(\bot)$$

Indukcija na m. Če $m=0$, dobimo na levi $\bot$ in je ✓

Denimo, da velja za m:

I.P. $\quad (\bigvee_i f_i)^m(\bot) \leq \bigvee_j \bigvee_k f_j^k(\bot)$ vemo

$$(\bigvee_i f_i)^{m+1}(\bot) = (\bigvee_i f_i)\left((\bigvee_i f_i)^m(\bot)\right) \leq (\bigvee_i f_i)\left(\bigvee_j \bigvee_k f_j^k(\bot)\right) =$$

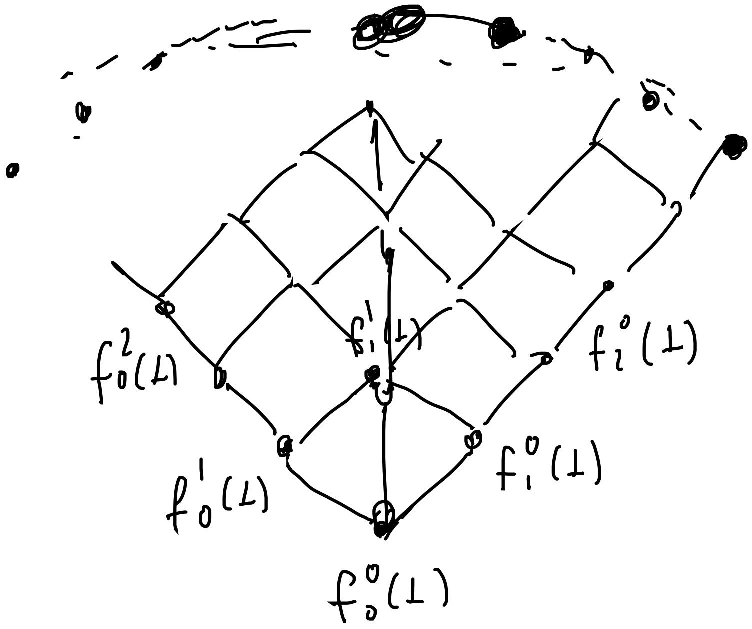
$$\bigvee_i \bigvee_j \bigvee_k f_i(f_j^k(\bot)) \underset{\text{?! zakaj}}{\leq} \bigvee_j \bigvee_k f_j^k(\bot)$$

bo objava

$$\bigvee_j \bigvee_k f_j^k(\perp)$$

$$\|$$

$$\bigvee_\ell f_\ell^\ell(\perp)$$



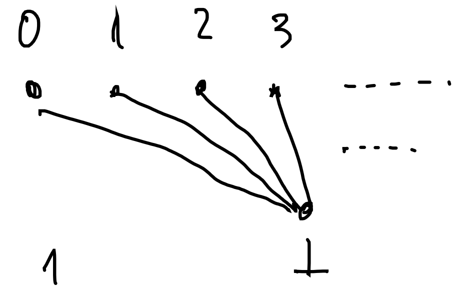
$$\bigvee_i \bigvee_j \bigvee_k f_i(f_j^k(\perp)) = \bigvee_p f_p(f_p^p(\perp)) = \bigvee_p f_p^{p+1}(\perp)) \leqslant$$

$$\bigvee_p f_{p+1}^{p+1}(\perp) = \bigvee_\ell f_\ell^\ell(\perp) \qquad \checkmark$$

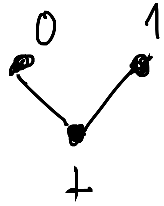
Semantika PCF

Ideja:
- tipi so domene
- izrazi so zvezne funkcije (izrazi v kontekstu)

Tip nat .... domena $\mathbb{N}_\perp$



Tip bool ...... domena $\{0,1\}_\perp$



$$[\![\mathrm{bool}]\!] = \{0,1\}_\perp$$

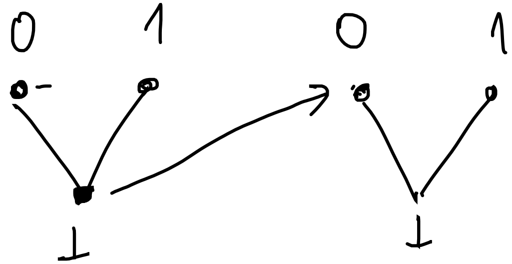
$$[\![\cdot \mid \mathrm{true}]\!] = 1$$

$$[\![\cdot \mid \mathrm{false}]\!] = 0$$

$$[\![\cdot \mid \mathrm{rec}\ x : \mathrm{bool}\ \mathrm{is}\ x]\!] = \perp$$

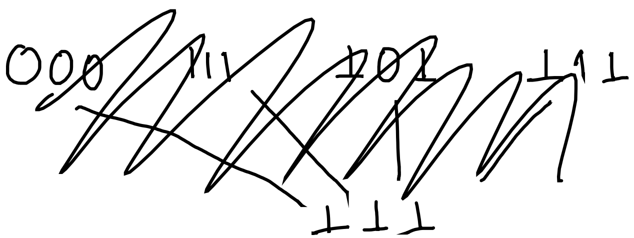
$$(\mathrm{rec}\ x : \mathrm{bool}\ \mathrm{is}\ \underline{x}) \longmapsto (\mathrm{rec}\ x : \mathrm{bool}\ \mathrm{is}\ x) \longmapsto \cdots$$

- $[\![\text{bool} \to \text{bool}]\!] = [\{0,1\}_\perp \to \{0,1\}_\perp]$

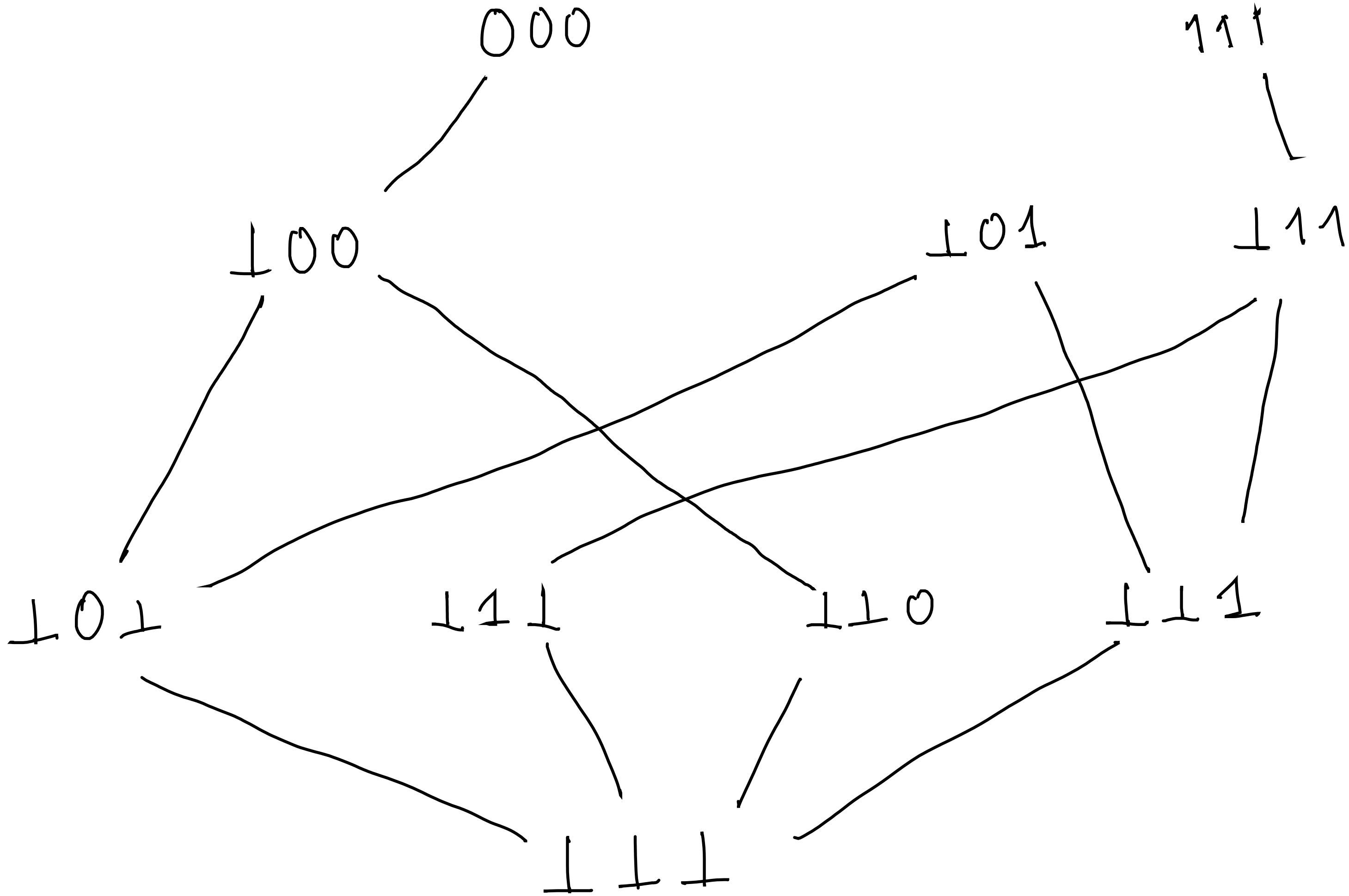


$f(\perp) = \perp$
$f(0) = 0$ zapišemo $\perp 00$
$f(1) = 0$

$f(\perp) = a$
$f(0) = b$ zapišemo abc
$f(1) = c$



$[\![\text{bool} \to \text{bool}]\!]$



$\cancel{\mathtt{true}\ e_2}$

$$\frac{e_1 \hookrightarrow \mathtt{false} \qquad e_3 \hookrightarrow v}{\mathtt{if}\ e_1\ \mathtt{then}\ e_2\ \mathtt{else}\ e_3 \hookrightarrow v}
\qquad\qquad
\frac{e_1 \hookrightarrow \mathtt{true} \qquad e_2 \hookrightarrow v}{\mathtt{if}\ e_1\ \mathtt{then}\ e_2\ \mathtt{else}\ e_3 \hookrightarrow v}$$

Relacija $\hookrightarrow$ med programi in vrednostmi

$$e \hookrightarrow v$$

program — vrednost

$\underbrace{\text{program} \qquad \text{vrednost}}_{\text{izrazi}}$

$$[\![e]\!] = x$$

izraz — element domene

Pomen fun

$x_1 : \tau_1, \ldots, x_n : \tau_n$ $\qquad \vec{t} = (t_1, \ldots, t_n) \in [\![\Gamma]\!]$ $\qquad$ element $[\![\tau \to \sigma]\!]$

$$[\![\Gamma \vdash (\mathsf{fun}\, x : \tau \to e) : \tau \to \sigma]\!](\vec{t}) = f$$

f neka zvezna funkcija.

$$f : [\![\tau]\!] \longrightarrow [\![\sigma]\!]$$

$$u \longmapsto [\![x : \tau, \Gamma \vdash e : \sigma]\!](u, t_1, \ldots, t_n)$$

$$[\![\Gamma \mid (\mathsf{fun}\ x : \tau \to e) : \tau \to \sigma]\!](\vec{t})(\underset{t_0}{u}) = [\![x : \tau, \Gamma \mid e : \sigma]\!]\underset{(t_0, \vec{t})}{(u, t_1, \ldots, t_n)}$$

Pomen rec

Ideja: $[\![\text{rec } x:\tau \text{ is } e]\!] =$

$[\![e[x/(\text{rec } x:\tau \text{ is } e)]]\!]$.

$$(\text{rec } x:\tau \text{ is } \phi(x)) = \phi(\text{rec } x:\tau \text{ is } \phi(x))$$

$\boxed{\begin{array}{l}\phi(x) = e \\ \phi = (x \mapsto e)\end{array}}$

$$[\![\Gamma \mid (\text{rec } x:\tau \text{ is } e):\tau]\!](\vec{t}) = \text{fix}_{[\![\tau]\!]}\ f$$

$\uparrow$?

$$f : [\![\tau]\!] \to [\![\tau]\!]$$
$$u \longmapsto [\![x:\tau, \Gamma \mid e:\tau]\!](u, \vec{t})$$

$$[\![\Gamma \mid (\text{rec } x:\tau \text{ is } e):\tau]\!](\vec{t}) = \text{fix}_{[\![\tau]\!]}\Big(\underset{t_0}{u} \mapsto [\![x:\tau, \Gamma \mid e:t]\!](\underset{t_0}{u}, \vec{t})\Big)$$