Univalence

① Path spaces – how to "compute" them?

type A, $x, y : A$

What does $x =_A y$ look like?

<u>Example</u> $\prod(x,y:\mathbb{1}).\ (x =_{\mathbb{1}} y) \simeq \mathbb{1}$. $\quad * : \mathbb{1}$

Construct $\quad f : \prod(x,y:\mathbb{1}).\ \overset{p}{x =_{\mathbb{1}} y} \rightarrow \overset{u}{\mathbb{1}}$

$g : \qquad \leftarrow$

$f\ x\ y\ p := *$

$g\ x\ y\ u : x =_{\mathbb{1}} y \qquad$ induction on $x : \mathbb{1}$ and $y : \mathbb{1}$

$g * * u : * =_{\mathbb{1}} *$

$:= \mathrm{refl}_*$

Verify: $\quad f\ x\ y\ (g\ x\ y\ u) = u$? $\quad$ for all $x, y, u : \mathbb{1}$

Induction on $u : \mathbb{1}$:

$$f\ x\ y\ (g\ x\ y\ *) \underset{\substack{\equiv \\ *}}{} = *$$

$f\ x\ y\ (g\ x\ y\ *) \equiv *$ by definition of f

Verify: $g\ x\ y\ (f\ x\ y\ p) = p$ for all $x, y : \mathbf{1}$, $p : x =_{\mathbf{1}} y$?

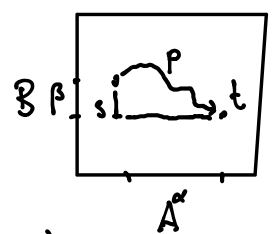
Induction on p: $x :\equiv z$, $y :\equiv z$, $p :\equiv \mathrm{refl}\ z$

$$g\ z\ z\ (f\ z\ z\ (\mathrm{refl}\ z)) = \mathrm{refl}\ z \ ? \quad \text{induction on } z : \mathbf{1}$$

$$g\ *\ *\ (f\ *\ *\ (\mathrm{refl}\ *)) \underset{\text{def of } g}{\equiv} \mathrm{refl}\ * \ \checkmark$$

$\square$

Example: $s, t : A \times B$



$$\prod(s,t : A \times B).\ (s =_{A \times B} t) \simeq (\pi_1 s =_A \pi_1 t) \times (\pi_2 s =_B \pi_2 t)$$

$$p : s = t \mapsto (\pi_1(p), \pi_2(p))$$

$$? \leftarrow (\alpha, \beta)$$

Example:

Recall: There is a map

$$X \cong Y \xrightarrow{\varepsilon} X \simeq Y.$$

If you just need any equivalence $e : X \simeq Y$, then it is enough to construct $i : X \cong Y$, then

$$e := \varepsilon(i).$$

Example: $f, g : A \to B$

$f =_{A\to B} g$ what it's like?

Would want to show:

$$(f =_{A\to B} g) \simeq \prod (x : A).\, fx =_B gx$$

$\neg X \equiv X \to 0$

Example: Say A has decidable equality when

$$\prod (x, y : A).\, (x =_A y) + \neg (x =_A y).$$

Theorem (Hedberg): if A has decidable equality then A is a set.

(2) What about the path spaces in a universe?

$A, B : \mathcal{U}$ What is $A =_{\mathcal{U}} B$ like?

$\{\text{Jake}\} = \{\text{pope}\}$

"Equivalent types are equal."

Want an equivalence

$$(A =_{\mathcal{U}} B) \simeq (A \simeq B)$$

"Equality is equivalent to equivalence".

$$\mathrm{idtoeq} : \prod(A,B:\mathcal{U}).\ A =_{\mathcal{U}} B \to A \simeq B$$

path induction:

$$\prod(C:\mathcal{U}).\ C \simeq C$$

$$C \mapsto \mathrm{ideq}_C \quad \text{the identity equivalence on } C$$

Given $p : A =_{\mathcal{U}} B$ we get $\mathrm{idtoeq}_{A,B}\, p : A \simeq B$.

<u>Univalence axiom (UA)</u>:

$$\prod(A,B:\mathcal{U}).\ \mathrm{isequiv}(\mathrm{idtoeq}_{A,B}).$$

This amounts to:

- there is a map

$$\mathrm{ua}_{A,B} : A \simeq B \longrightarrow A =_{\mathcal{U}} B$$

- there is a map

$$\mathrm{idtoeq}_{A,B} : A =_{\mathcal{U}} B \longrightarrow A \simeq B$$

- Inverses:

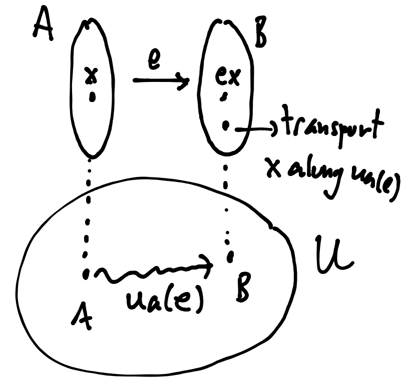
$$\mathrm{ua}(\mathrm{idtoeq}\, p) =_{A =_{\mathcal{U}} B} p$$

$$\mathrm{idtoeq}(\mathrm{ua}(e)) =_{A \simeq B} e$$

- Computation rule:

$$\mathrm{transport}^{(\lambda X:\mathcal{U}.X)}(\mathrm{ua}(e), x) =_B ex$$

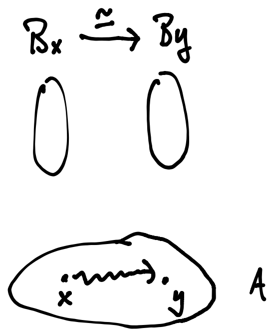
$e : A \simeq_{\mathcal{U}} B$



In general:

$B : A \to \mathcal{U}$ is <u>univalent</u> when

$$\prod(x, y : A).\ (x =_A y) \simeq (Bx \simeq By)$$



When is $B : 1 \to \mathcal{U}$ univalent?

$$\prod(x, y : 1).\ (x =_1 y) \simeq (Bx \simeq By)$$

$$(* =_1 *) \simeq (B* \simeq B*)$$

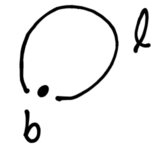
$$1 \simeq (B* \simeq B*)$$

Plug in $Id : \mathcal{U} \to \mathcal{U}$ to get UA.

3 $\pi_1(S^1) = \mathbb{Z}$

Recall:

- circle S^1 $\quad b : S^1$ base
  $\quad \ell : b = b$ loop



- $(\mathbb{Z}, +, 0)$ the additive group of integers

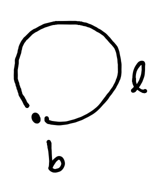
- fundamental group:
  A, $a : A$ $\quad$ loop space at a:
  $$\Omega(A, a) :\equiv (a =_A a)$$
  $$\pi_1(A, a) :\equiv \| \Omega(A, a) \|_0$$

We show that $\Omega(S^1, b) = \mathbb{Z}$, so it follows

$$\pi_1(S^1, b) = \| \Omega(S^1, b) \|_0 = \| \mathbb{Z} \|_0 = \mathbb{Z}$$

Enough to show $\Omega(S^1, b) \simeq \mathbb{Z}$. by UA.

$$(b =_{S^1} b) \simeq \mathbb{Z}$$



$f : \mathbb{Z} \to (b =_{S^1} b)$

$g : (b =_{S^1} b) \to \mathbb{Z}$

Candidate for f:

$$f(k) := \begin{cases} \underbrace{\mathrm{loop} \cdot \mathrm{loop} \cdot \ldots \cdot \mathrm{loop}}_{k} & \text{if } k > 0 \\ \mathrm{refl}_b & \text{if } k = 0 \\ \underbrace{\mathrm{loop}^{-1} \cdot \ldots \cdot \mathrm{loop}^{-1}}_{-k} & \text{if } k < 0 \end{cases} =: \mathrm{loop}^k$$

Candidate for g:

$$g(p) := \ ? \text{ winding number}$$

Verify: $\prod (p : b = b).\ f(g(p)) =_{b=b} p$.

Problem: Cannot use <u>path-induction</u> on p.

Define:
$$C : S^1 \to \mathcal{U}$$
$$D : S^1 \to \mathcal{U}$$

$$\boxed{\begin{array}{l} S^1 \to X \\ b \mapsto x \\ \ell \mapsto p : x =_X x \end{array}}$$

as follows:

- Define D: $\quad D(x) := (b =_{S^1} x) \quad$ universal cover

- Define C by <u>circle</u> induction: $\quad$ "deck transformations"

$$C(b) := \mathbb{Z}$$

$$C(\mathrm{loop}) : C(b) =_{\mathcal{U}} C(b)$$

$$C(\mathrm{loop}) : \mathbb{Z} =_{\mathcal{U}} \mathbb{Z}$$

$\vdots$

$\Vert$

$$\mathrm{ua}(\mathrm{succ}) : \mathbb{Z} =_{\mathcal{U}} \mathbb{Z} \quad \text{because} \quad \begin{array}{l} \mathrm{succ}(k) = k+1 \\ \mathrm{pred}(k) = k-1 \end{array}$$

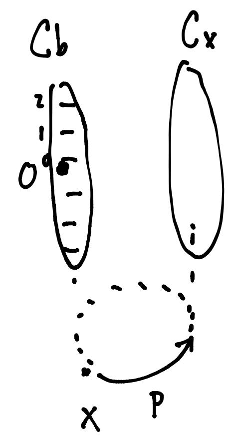
$$\mathrm{succ} : \mathbb{Z} \simeq \mathbb{Z}$$

<u>Plan:</u> $\prod(x : S^1) . \; Dx \simeq Cx$

$$\prod(x : S^1) . \; (b =_{S^1} x) \simeq Cx$$

In particular when $x \equiv b$, we get

$$(b =_{S^1} b) \simeq \mathbb{Z}$$



Define: $e : \prod(x : S^1) . \; Dx \to Cx$

$$e \; x \; p := \mathrm{transport}^{C}(p, 0)$$

$p : b =_{S^1} x$

Define: $d : \prod(x : S^1) . \; Cx \to Dx$

by circle induction

$$d(b) : Cb \to Db$$
$$\mathbb{Z} \to b = b$$

$$d(b) := \lambda k : \mathbb{Z} . \; \mathrm{loop}^{k}$$

Also need to lift the loop: (find a path above the loop)

SKIP

Verify:

$$d\,x\,(e\,x\,p) = p\ ?$$

$x : S^1$

$p : b =_{S^1} x$

for all $x : S^1$, all $p : b = x$. Path induction on p:

$$d\,b\,(e\,b\,(\mathrm{refl}\,b)) =$$

$$d\,b\,(\mathrm{transport}^C(\mathrm{refl}\,b, 0)) =$$

$$d\,b\,0 = \mathrm{loop}^0 = \mathrm{refl}\,b \quad \checkmark$$

Verify: $e\,x\,(d\,x\,c) = c$ for all $x : S^1$, $c : C x$

Circle induction: verify for $x \equiv b$

$c : C b \equiv \mathbb{Z}$

$$e\,b\,(d\,b\,c) = c\ ?$$

for all c

$$\begin{aligned} e\,b\,(d\,b\,c) &= e\,b\,(\mathrm{loop}^c) \\ &= \mathrm{transport}^C(\mathrm{loop}^c, 0) \\ &= \mathrm{succ}^c(0) \\ &= c \end{aligned}$$