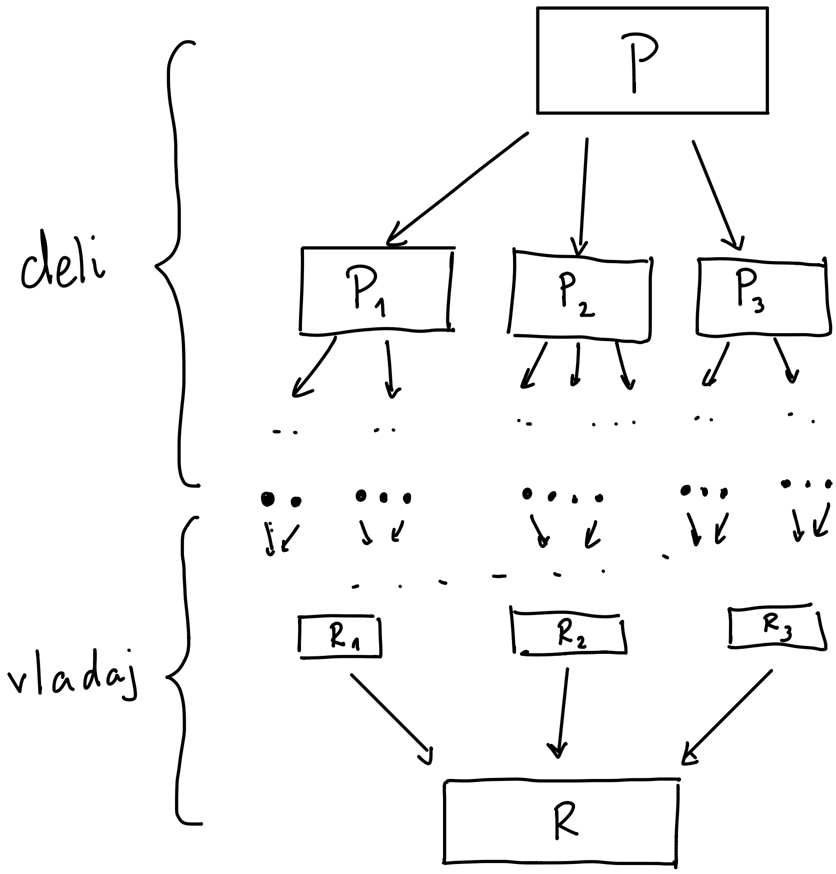
Deli & vladaj



1) podproblemi $P_1, P_2, \ldots$ so istega tipa kot P

2) podprobleme rešimo z istim postopkom, rekurzivno

← osnovni problemi — rešimo neposredno (jih ne delimo)

Primeri

- <u>Urejanje z zlivanjem</u> (merge sort)
  - problem: uredi tabelo (dolžine n)
  - deli : razdeli tabelo na levo in desno polovico (2 podproblema) velikosti $\frac{n}{2}$
  - vladaj : zlij urejeni tabeli

<u>Bisekcija</u>

- problem : poišči indeks elementa x v urejeni tabeli (dolžine n)
- deli : išči v levi ali desni podtabeli (1 podproblem, velikost $\frac{n}{2}$)
- vladaj : vrni odgovor

Strassenovo množenje matrik

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad n \times n \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} \quad n \times n$$

$$C = A \cdot B = \begin{bmatrix} c_{11} & c_{1n} \\ c_{n1} & c_{nn} \end{bmatrix} \qquad c_{i,j} = \sum_{k=1}^{n} a_{i,k} \cdot b_{k,j}$$

Naivni postopek množenja:

množenje: n^2 elementov matrike $\times$ n množenj $= n^3$ množenj $\Big\}$ $O(n^3)$ operacij

seštevanje: $n^2 \times (n-1)$ $= O(n^3)$ seštevanj

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for i in range(n):
    for j in range(n):
        for k in range(n):
            O(1)
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$\Big\}$ $O(n^3)$

Seštevanje: $D = A + B$ $\quad d_{i,j} = a_{i,j} + b_{i,j}$ $\quad O(n^2)$ operacij

Pozor: vhodni podatki so velikosti $2n^2$

seštevanje matrik je učinkovito (linearni čas)

Bločno množenje

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \qquad B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

$$C = \left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right] \qquad \begin{aligned} C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\ &\vdots \\ C_{22} &= \ldots \end{aligned}$$

To je deli in vladaj! Ali se splača tako delati?

Časovna zahtevnost postopka deli & vladaj:

- problem P velikosti n
- deli:
  - razdelimo na k podproblemov velikosti $\alpha \cdot n$, $0<\alpha<1$
  - opravimo $f(n)$ korakov
- vladaj:
  - opravimo $g(n)$ korakov
- osnovni problem : velikost 1, 1 korak

$T(n)$ = časovna zahtevnost celotnega postopka za problem velikosti n

Vemo:

$$T(n) = f(n) + k \cdot T(\alpha \cdot n) + g(n)$$
$$T(1) = 1$$

Obravnavajmo primer $f(n) + g(n) = n^m$ korakov

Dobimo:

$$T(n) = n^m + k \cdot T(\alpha n) \qquad T(1) = 1$$

$$\begin{aligned}
T(n) &= n^m + k \cdot T(\alpha n) = \\
&= n^m + k \cdot ((\alpha n)^m + k \cdot T(\alpha^2 n)) = \\
&= n^m(1 + k \cdot \alpha^m) + k^2 T(\alpha^2 n) = \\
&= n^m(1 + k \cdot \alpha^m) + k^2(\alpha^{2m} \cdot n^m + kT(\alpha^3 n)) = \\
&= n^m(1 + k\alpha^m + k^2\alpha^{2m}) + k^3 T(\alpha^3 n) = \\
&\ \vdots \\
&= n^m(1 + k\alpha^m + \cdots + k^j\alpha^{jm}) + k^j T(\underbrace{\alpha^j n}_{=1?})
\end{aligned}$$

$$\alpha^j n = 1$$
$$\alpha^j = \frac{1}{n}$$
$$j = \log_\alpha \frac{1}{n} = -\log_\alpha n = \log_{1/\alpha} n$$

$$= n^m \cdot \underbrace{\sum_{i=0}^{\log_{1/\alpha} n} (k\alpha^m)^i}_{\text{geometrijska}} + k^{\log_{1/\alpha} n} \cdot 1$$

$$\sum_{i=0}^{\ell} r^i = \frac{1-r^{\ell+1}}{1-r}$$

$$\cancel{x^{\log_b y} = \left(b^{\log_b x}\right)^{\log_b y}}$$

$$= n^m \cdot \frac{1}{1-k\alpha^m} \cdot \left(1-(k\alpha^m)^{1+\log_{1/\alpha} n}\right) + k \cdot \log_{1/\alpha} n$$

Pišemo $r = k\alpha^m$, $\beta = 1/\alpha$

$$= n^m \cdot \frac{1}{r-1} \cdot \left(r^{1+\log_\beta n} - 1\right) + k \cdot \log_\beta n$$

$$= \frac{r}{r-1} \cdot n^m \cdot r^{\log_\beta n} - \frac{1}{r-1} \cdot n^m + k \cdot \log_\beta n$$

Pogoj: $r-1>0$

$r > 1 \iff k \cdot \alpha^m > 1$

<u>Rezultat:</u> deli & vladaj

- k podproblemov velikosti $\alpha \cdot n$, $0<\alpha<1$
- v vsaki fazi skupaj $O(n^m)$ dela
- smiselno, če je $r := k \cdot \alpha^m > 1$

$$T(n) \in O\left(n^m \cdot r^{\log_{1/\alpha} n} + n^m + k \cdot \log_{1/\alpha} n\right)$$

<u>Izboljšava</u>: namesto faktorja α uporabimo $\beta = 1/\alpha$:

- k podproblemov velikosti $\frac{n}{\beta}$, $\beta > 1$
- v vsaki fazi $O(n^m)$ dela
- <u>smiselno, če je $k > \beta^m$</u> ?

$$T(n) \in O\left(n^m \cdot \left(\frac{k}{\beta^m}\right)^{\log_\beta n} + k \cdot \log_\beta n\right)$$

<u>Primeri:</u>

- urejanje z zlivanjem: $k=2$, $\beta=2$, $m=1$

Poseben primer: $r = 1$

$$T(n) = n^m \cdot \sum_{i=0}^{\log_{1/\alpha} n} (\underbrace{k\alpha^m}_{1})^i + k^{\log_{1/\alpha} n} \cdot 1$$

$$= n^m \cdot \log_\beta n + k \cdot \log_\beta n$$

$$\boxed{\text{Če je } k = \beta^m: \quad T(n) \in O(n^m \cdot \log_\beta n)}$$

Torej, urejanje z zlivanjem je $O(n \cdot \log_2 n)$

- Bisekcija: $k=1,\ \beta=2,\ m=0 \Rightarrow O(\log_2 n)$

- Bločno množenje matrik

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \qquad B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

$$C = \left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right]$$

$$\begin{aligned}
\mathbf{C}_{1,1} &= \mathbf{A}_{1,1}\mathbf{B}_{1,1} + \mathbf{A}_{1,2}\mathbf{B}_{2,1} \\
\mathbf{C}_{1,2} &= \mathbf{A}_{1,1}\mathbf{B}_{1,2} + \mathbf{A}_{1,2}\mathbf{B}_{2,2} \\
\mathbf{C}_{2,1} &= \mathbf{A}_{2,1}\mathbf{B}_{1,1} + \mathbf{A}_{2,2}\mathbf{B}_{2,1} \\
\mathbf{C}_{2,2} &= \mathbf{A}_{2,1}\mathbf{B}_{1,2} + \mathbf{A}_{2,2}\mathbf{B}_{2,2}
\end{aligned}$$

$K = 8, \quad \beta = 2, \quad m = 2$ $\qquad O\left(n^m \cdot \left(\frac{k}{\beta^m}\right)^{\log_\beta n} + k \cdot \log_\beta n\right)$

$$O(n^2 \cdot 2^{\log_2 n} + 8 \cdot \log_2 n) =$$

$$O(n^3)$$

Bločno množenje je še vedno $O(n^3)$

Strassen:

$$
\begin{aligned}
\mathbf{M}_1 &:= (\mathbf{A}_{1,1} + \mathbf{A}_{2,2})(\mathbf{B}_{1,1} + \mathbf{B}_{2,2}) \\
\mathbf{M}_2 &:= (\mathbf{A}_{2,1} + \mathbf{A}_{2,2})\mathbf{B}_{1,1} \\
\mathbf{M}_3 &:= \mathbf{A}_{1,1}(\mathbf{B}_{1,2} - \mathbf{B}_{2,2}) \\
\mathbf{M}_4 &:= \mathbf{A}_{2,2}(\mathbf{B}_{2,1} - \mathbf{B}_{1,1}) \\
\mathbf{M}_5 &:= (\mathbf{A}_{1,1} + \mathbf{A}_{1,2})\mathbf{B}_{2,2} \\
\mathbf{M}_6 &:= (\mathbf{A}_{2,1} - \mathbf{A}_{1,1})(\mathbf{B}_{1,1} + \mathbf{B}_{1,2}) \\
\mathbf{M}_7 &:= (\mathbf{A}_{1,2} - \mathbf{A}_{2,2})(\mathbf{B}_{2,1} + \mathbf{B}_{2,2})
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{C}_{1,1} &= \mathbf{M}_1 + \mathbf{M}_4 - \mathbf{M}_5 + \mathbf{M}_7 \\
\mathbf{C}_{1,2} &= \mathbf{M}_3 + \mathbf{M}_5 \\
\mathbf{C}_{2,1} &= \mathbf{M}_2 + \mathbf{M}_4 \\
\mathbf{C}_{2,2} &= \mathbf{M}_1 - \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_6
\end{aligned}
$$

$O(n^m \cdot (\frac{k}{\beta^m})^{\log_\beta n} + k \cdot \log_\beta n)$

$k = 7, \quad \beta = 2, \quad m = 2$

$$O\left(n^2 \cdot \left(\tfrac{7}{4}\right)^{\log_2 n} + 7 \cdot \log_2 n\right) =$$

$$O\left(n^2 \cdot n^{\log_2 \frac{7}{4}}\right) =$$

$$O\left(n^{\log_2 7}\right) \subseteq O\left(n^{2.81}\right)$$

$\frac{7}{4} = 2^{\log_2 \frac{7}{4}}$

$\left(2^{\log_2 \frac{7}{4}}\right)^{\log_2 n} =$

$n^{\log_2 \frac{7}{4}}$

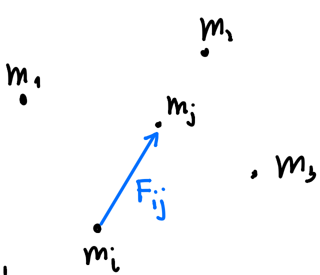
$\log_2 \frac{7}{4} = \log_2 7 - 2$

Barnes-Hutov algoritem za simulacijo gravitacije

<u>Naloga</u>: → n teles z masami $m_1, \ldots, m_n$ in pozicijami $\vec{x}_1, \ldots, \vec{x}_n$

→ izračunaj pospeške na telesa pod vplivom gravitacije

Naivna rešitev:



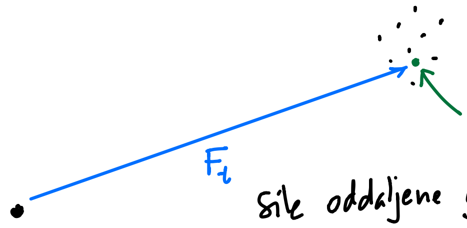
izračunamo vse sile na i-to telo:

$$\vec{F}_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \vec{F}_{ij}$$

skupaj $O(n^2)$ sil

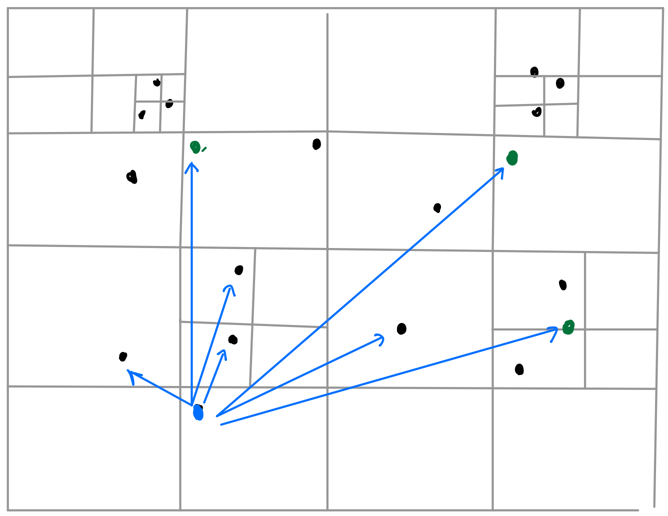
naivni algoritem $O(n^2)$.

Ideja

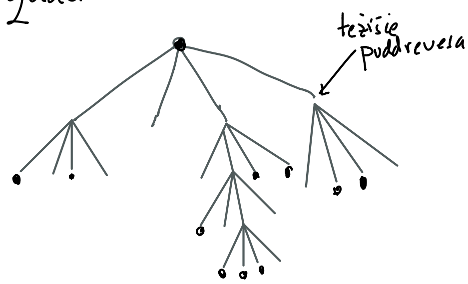


sile oddaljene gruče nadomestimo s silo enega telesa v težišču (s skupno maso)

Deli in vladaj:



"quad tree"



$O(n \cdot \log n)$