Two Constructive Embedding-Extension Theorems and Applications

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Alex Simpson says “hi”

Contribution

Two constructive embedding-extension theorems:

1. Baire space into locally non-compact spaces
2. Cantor space into spaces without isolated points

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Two constructive embedding-extension theorems:

1. Baire space into locally non-compact spaces
2. Cantor space into spaces without isolated points

This gives a method for transferring properties of Baire space to locally non-compact spaces. Two applications:

1. Relationship between choice and continuity principles
2. Banach-Mazur computability

Overview

1. Setup & definitions
2. The embedding-extension theorems
3. Application to choice & continuity
4. Application to Banach-Mazur computability

Constructive setting

We work in Bishop-style constructive mathematics.
(Arguments are also valid in topos logic.)

We use countable choice:

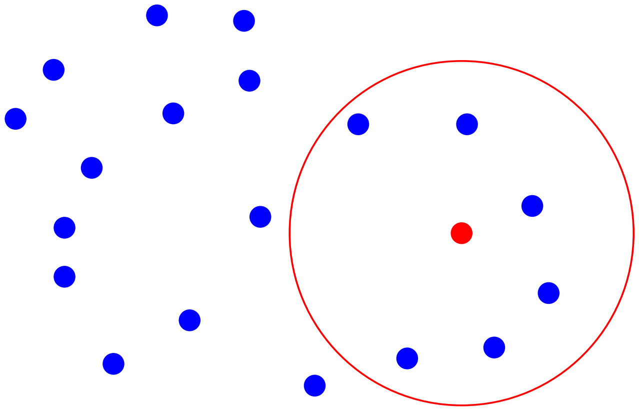
$$(\forall\, n \in \mathbb{N} \,.\, \exists\, x \in X \,.\, \varphi(n, x)) \implies \exists\, f \in X^{\mathbb{N}} \,.\, \forall\, n \in \mathbb{N} \,.\, \varphi(n, f(n))$$

We do not use dependent choice.

We use constructive notions of *metric space*, *complete separable (CSM)*, *complete totally bounded (CTB)*, …

Locally non-compact space

A sequence $\langle a_i \rangle_i$ in X is *without accumulation point* when for every $x \in X$ there are $\epsilon > 0$ and $m \in \mathbb{N}$ such that $d(x, a_n) > \epsilon$ for all $n \geq m$.



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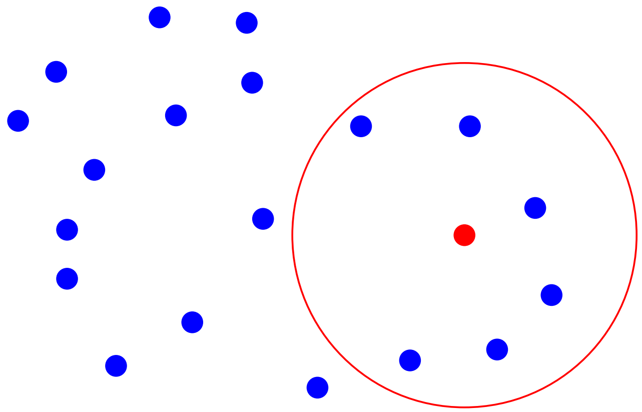

A space X is *locally non-compact* if every open ball contains a sequence without accumulation point.

Examples of locally non-compact spaces

1. Infinite-dimensional metric vector spaces:

$$\ell^2, \quad \mathrm{L}^2[0,1], \quad \mathcal{C}_{\mathrm{u}}([0,1]), \quad \mathbb{R}^{\mathbb{N}}, \quad \ldots$$

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3. A surprising example in recursive mathematics: a *strong Specker sequence* is a sequence without accumulation point. It follows that $[0,1]$ is locally non-compact in the setting of recursive mathematics (in the effective topos).

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3. if f is pointwise continuous then so is $\overline{f}$.

Proof of extension-embedding theorem

$$\begin{tikzcd}
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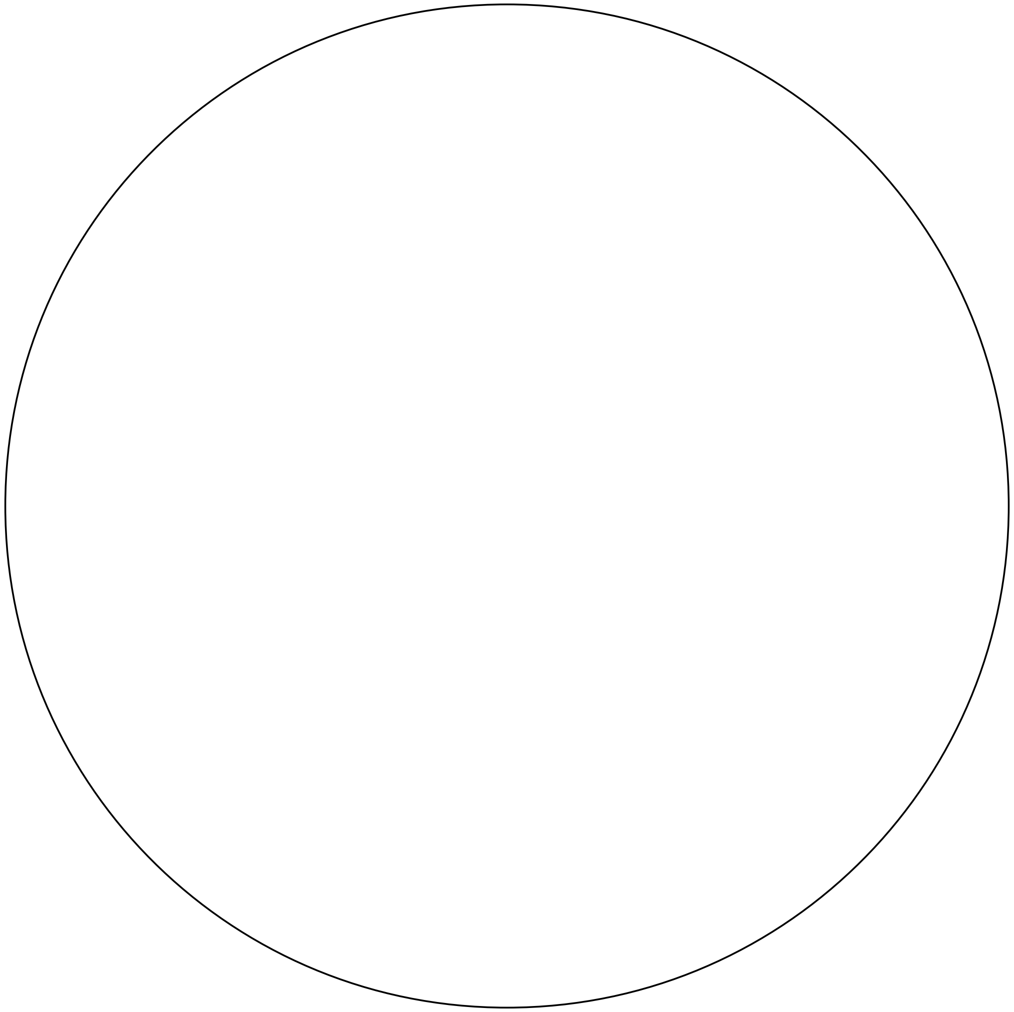
2. Construct h from a given f.
3. Take $\overline{f} = h \circ g$.

Embedding $\mathbb{Z}^{\mathbb{N}}$ into X



Start with a ball.

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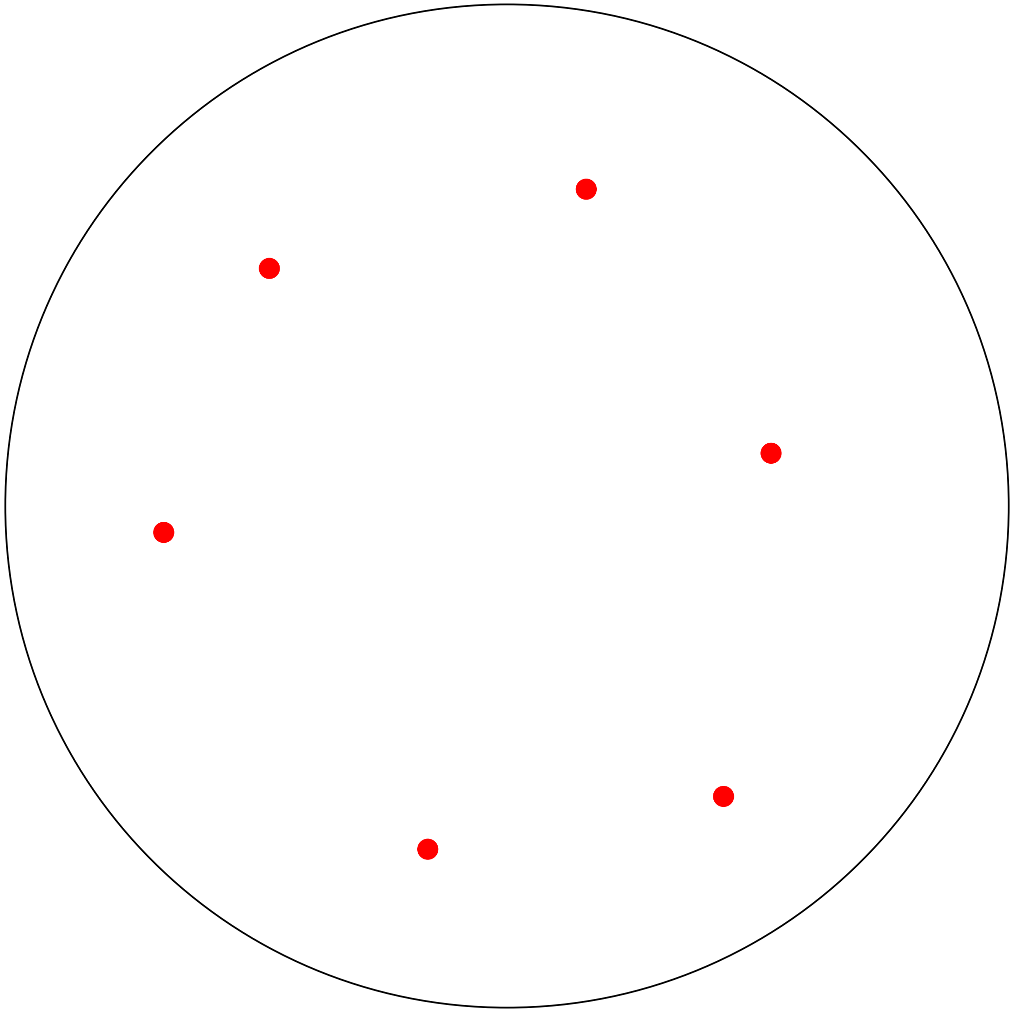

Find a sequence $\langle a_i \rangle_i$ without accumulation point.

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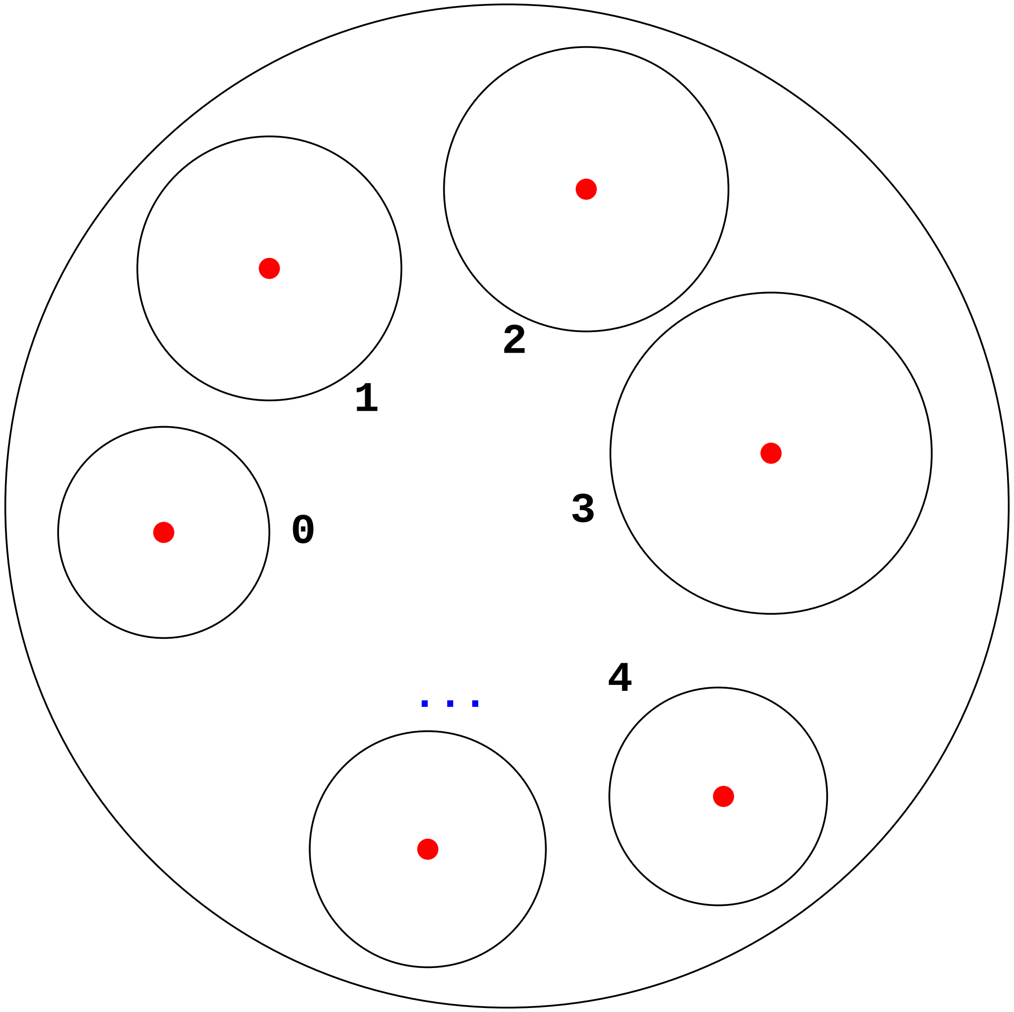

Find disjoint balls with centers at a_i's.

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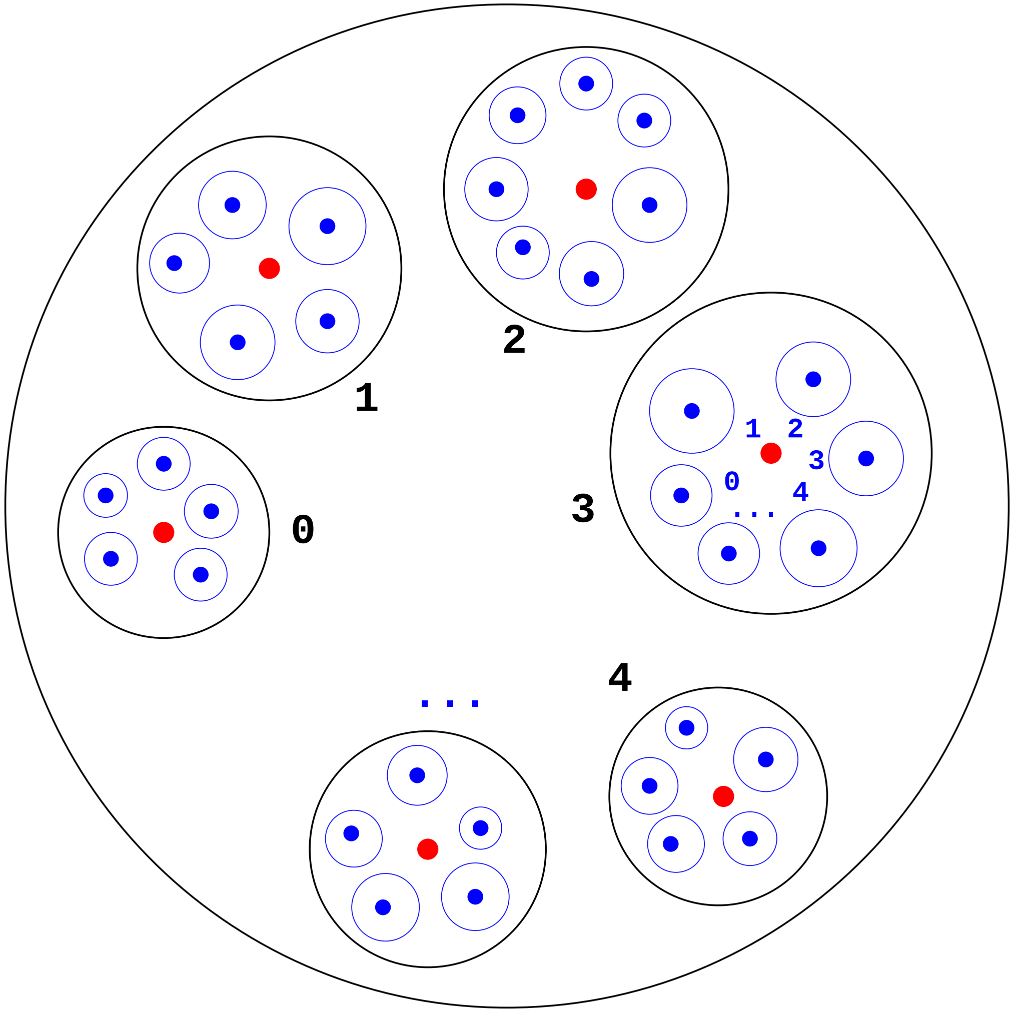

Repeat the construction inside the balls to get sequences $\langle a_i \rangle_i$, $\langle a_{i,j} \rangle_j$, $\langle a_{i,j,k} \rangle_k$, $\ldots$

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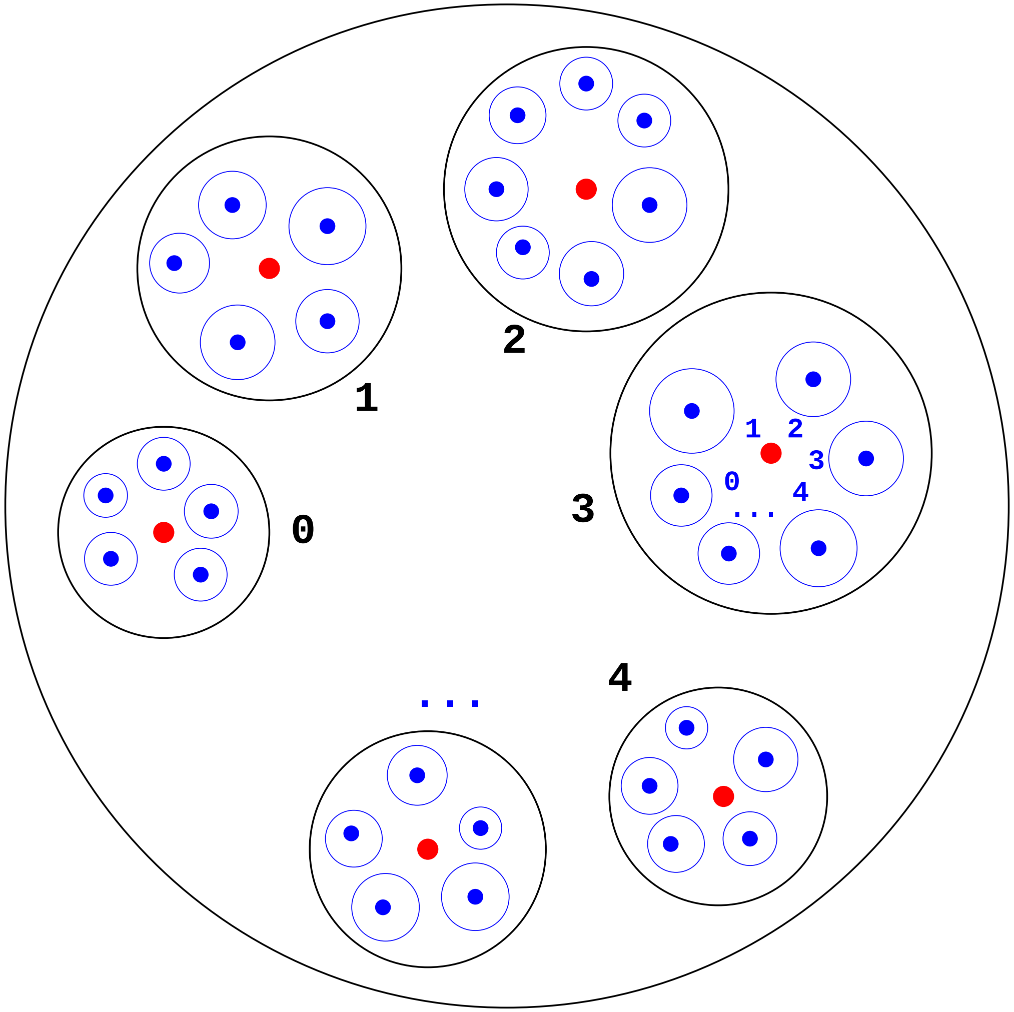
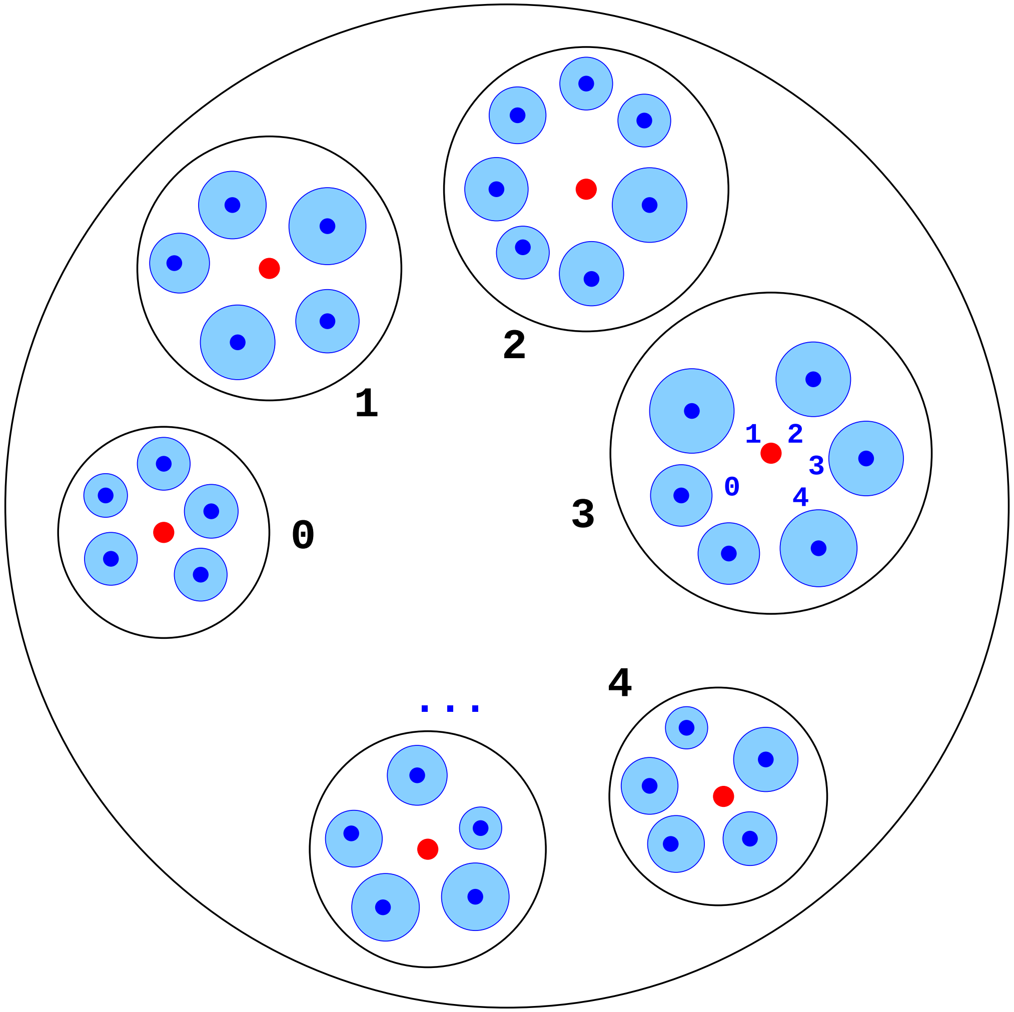


Define $e(\alpha) = \lim_{n\to\infty} a_{\alpha_0,\alpha_1,\ldots,\alpha_n}$.

Construction of $g : X \to \mathbb{R}^{\mathbb{N}}$



$g(x)_i = j$ in the j-th ball of level i;

$g(x)_i = 0$ outside balls of level $i - 1$.

Construction of $h : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ from $f : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$

For $\gamma \in \mathbb{R}^{\mathbb{N}}$, find $\beta \in \mathbb{Z}^{\mathbb{N}}$ *adequate* for γ:

$$|\beta_n - \gamma_n| < 2/3$$

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$$\begin{aligned} h_0^\beta(\gamma) &= f(0^\omega) \\ h_{i+1}^\beta(\gamma) &= (1 - w_{i,\beta,\gamma}) \cdot h_i^\beta(\gamma) + w_{i,\beta,\gamma} \cdot f(\beta_0 \ldots \beta_i 0^\omega)\ . \end{aligned}$$

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Here $0 \leq w_{i,\beta,\gamma} \leq 1$ is a weight function such that:

1. $w_{i,\beta,\beta} = 1$
2. if β', β both adequate for γ differ in j-th term then $w_{i,\beta,\gamma} = 0$ for $i \geq j$.

Spaces without isolated points

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Examples of spaces without isolated points:
$\mathbb{R}$, $\mathbb{Q}$, locally non-compact spaces, Cantor space:

$$2^{\mathbb{N}} = \left\{\alpha \in \mathbb{Z}^{\mathbb{N}} \mid \forall\, n \in \mathbb{N} \,.\, 0 \leq \alpha_n \leq 1\right\} \ .$$

Embedding-Extension Theorem for $2^{\mathbb{N}}$

For any inhabited CSM X without isolated points:

1. there is a uniformly continuous embedding $e : 2^{\mathbb{N}} \to X$ with closed image,
2. every sequentially continuous $f : 2^{\mathbb{N}} \to \mathbb{Z}$ extends to sequentially continuous $\overline{f} : X \to \mathbb{R}$,

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Choice and Continuity Principles

Choice axiom $\mathsf{AC}(X, Y)$:

$$(\forall\, x \in X \,.\, \exists\, y \in Y \,.\, \varphi(x, y)) \implies \exists\, f \in Y^X \,.\, \forall\, x \in X \,.\, \varphi(x, f(x))$$

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How do they interact?

$\mathsf{AC}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$ and continuity principles

If $\mathsf{AC}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$ holds then:

1. $\mathsf{CP}(\mathbb{Z}^{\mathbb{N}}, \mathbb{N})$ implies $\mathsf{CP}(X, Y)$ for all CSM X and separable Y.

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3. Let $\mathbb{N}^+$ be the one-point compactification of $\mathbb{N}$. Then $\mathsf{CP}(\mathbb{N}^+, \mathbb{N})$ implies "all maps $X \to Y$ are sequentially continuous", for all CSMs X and Y.

$\mathsf{AC}(\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}, \mathbb{N})$ and continuity principles

If $\mathsf{AC}(\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}, \mathbb{N})$ holds then *not* $\mathsf{CP}(\mathbb{Z}^{\mathbb{N}}, \mathbb{N})$.

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Theorem: $[\mathsf{CP}(\mathbb{N}^{+}, \mathbb{N})]$

1. If X is inhabited CSM without isolated points and $\mathsf{CP}(X, \mathbb{R})$ then $\mathsf{CP}(\mathsf{2}^{\mathbb{N}}, \mathbb{N})$.
2. If X is inhabited locally non-compact CSM and $\mathsf{CP}(X, \mathbb{R})$ then $\mathsf{CP}(\mathbb{Z}^{\mathbb{N}}, \mathbb{N})$.

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Corollary: $[\mathsf{CP}(\mathbb{N}^{+}, \mathbb{N})]$
If $\mathsf{AC}(\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}, \mathbb{N})$ and X is inhabited locally non-compact CSM then *not* $\mathsf{CP}(X, \mathbb{N})$.

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A map $f : X \to Y$ is *Banach-Mazur computable (BM-computable)* if it maps computable sequences in X to computable sequences in Y.

Clearly, every computable $f : X \to Y$ is BM-computable.

Does the converse also hold?

BM-computable but not computable maps

Theorem: [Friedberg ’58]
There exists a BM-computable $f : \mathbb{Z}^{\mathbb{N}} \to \mathbb{N}$ which is not computable. (There is also one $f : 2^{\mathbb{N}} \to \mathbb{Z}$.)

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Theorem:

If X is an inhabited computable CSM without isolated points then there exists a BM-computable $f : X \to \mathbb{R}$ which is not computable.

Proof of Theorem

For inhabited X without isolated points, we prove that every BM-computable $f : 2^{\mathbb{N}} \to \mathbb{Z}$ extends to BM-computable $h : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$:

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Now take $f : 2^{\mathbb{N}} \to \mathbb{Z}$, BM-computable but not computable.
Then $\overline{f} = h \circ g$ is BM-computable.
But if $\overline{f}$ were computable then so would be f.